

The Eigenvalue Sum for a One-Dimensional Potential

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Contents

Introduction.

Review of earlier results.

Truncated eigenvalue sums.

The First WKB Eigenvalue Sum Theorem.

Low eigenvalues in a potential with a Coulomb singularity.

The Second WKB Eigenvalue Sum Theorem.

Eigenvalue sums for degenerate potentials.

The Third WKB Eigenvalue Sum Theorem.

INTRODUCTION

This paper is part of a series [FS2–FS7] proving the asymptotic formula announced in [FS1] for the ground-state energy of an atom. Our goal here is to give precise estimates for the sum of the negative eigenvalues of an ordinary differential operator

$$H = -\frac{d^2}{dx^2} + V(x) \quad \text{on an interval } I_{\text{BVP}}. \quad (1)$$

We denote this sum by $\text{sneg}(H)$.

The potentials of interest to us are large and slowly varying. A basic example is

$$V(x) = \lambda^2 V_1(x) \quad \text{on } I_{\text{BVP}} = [-1, 1], \quad (2)$$

with V_1 a fixed smooth function and λ a large parameter. We suppose

$$V_1(0) < 0, \quad V_1'(0) = 0, \quad (3)$$

* Partially supported by the NSF under Grant DMS-9104455.A01.

$$V''_1 > c > 0 \text{ on } [-1, 1], \quad \{V_1 < 0\} \subset (-1, 1).$$

For such potentials, a standard approximate formula for the eigenvalue sum is

$$\text{sneg}(H) \approx -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx, \quad (4)$$

where $t_+^s = t^s$ if $t > 0$, $t_+^s = 0$ if $t \leq 0$.

Unfortunately, (4) is too crude for our purposes. To give a sharper approximation, we set

$$\phi(0) = \int_{I_{\text{BVP}}} (-V(x))_+^{1/2} dx, \quad \phi'(0) = \frac{1}{2} \int_{I_{\text{BVP}}} (-V(x))_+^{-1/2} dx, \quad (5)$$

and

$$\tilde{\chi}(t) = \min_{k \in \mathbb{Z}} \left\{ |t - (k + \tfrac{1}{2})|^2 - \tfrac{1}{12} \right\} \quad \text{for } t \in \mathbb{R}. \quad (6)$$

Our basic result for potentials of the form (2), (3) is as follows.

THEOREM 1. *The sum of the negative eigenvalues of H is given by*

$$\begin{aligned} \text{sneg}(H) = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) (-V(x))_+^{-1/2} dx \\ & + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \text{Error}, \end{aligned} \quad (7)$$

with $|\text{Error}| < C_\varepsilon \lambda^\varepsilon$. Here, C_ε may depend on $\varepsilon > 0$ and on V_1 , but not on λ .

On the right in (7), the first term has size $\sim \lambda^3$, while the next two terms are $\sim \lambda$.

To motivate Theorem 1, we recall the textbook derivation of (4). At the heart of the argument is the WKB approximation (see Erdélyi [E]). According to WKB the eigenvalues E_k of H satisfy

$$\phi(E_k) \approx \pi(k + \tfrac{1}{2}) \quad \text{for integers } k, \quad (8)$$

with

$$\phi(E) = \int_{I_{\text{BVP}}} (E - V(x))_+^{1/2} dx. \quad (9)$$

Defining $E(t)$ as the solution of $\phi(E) = \pi(t + \frac{1}{2})$, we rewrite (8) as $E_k \approx E(k)$, which gives

$$\text{sneg}(H) \approx \sum_{k \in [a, b] \cap \mathbb{Z}} E(k), \quad \text{with } a = \frac{1}{\pi} \phi(\min V) - \frac{1}{2}, \quad b = \frac{1}{\pi} \phi(0) - \frac{1}{2}. \quad (10)$$

It is reasonable to guess that

$$\sum_{k \in [a, b] \cap \mathbb{Z}} E(k) \approx \int_a^b E(t) dt. \quad (11)$$

Elementary manipulations show that $\int_a^b E(t) dt = -(2/3\pi) \int_{\text{BVP}} (-V(x))_+^{-3/2} dx$. Hence, the usual approximation (4) for $\text{sneg}(H)$ follows from (8) and (11).

Our previous papers [FS2, FS3] prove refinements of the basic approximations (8) and (11). Specifically, the eigenvalues E_k satisfy

$$\phi(E_k) + \frac{1}{48} \psi(E_k) \approx \pi(k + \tfrac{1}{2}), \quad (12)$$

where

$$\psi(E) = \lim_{\delta \rightarrow 0+} \left\{ \int_{E - V(x) > \delta} V''(x) (E - V(x))_+^{-3/2} dx - q(E) \delta^{-1/2} \right\} \quad (13)$$

and $q(E)$ is defined to make the limit finite in (13).

Equation (8) holds modulo errors $O(\lambda^{-1})$, while (12) holds modulo errors $O(\lambda^{\varepsilon-2})$. (See [FS2].) Regarding (11), we studied $\sum_{k \in \mathbb{Z} \cap [a, b]} f(k)$ for general, slowly varying functions f . We showed in [FS3] that

$$\begin{aligned} \sum_{k \in \mathbb{Z} \cap [a, b]} f(k) &\approx \int_a^b f(t) dt - f(b) \chi_-(b) - f(a) \chi_+(a) \\ &\quad + \frac{1}{2} f'(b) \tilde{\chi}(b) - \frac{1}{2} f'(a) \tilde{\chi}(a) \end{aligned} \quad (14)$$

with

$$\chi_-(t) = (t - k - \tfrac{1}{2}) \quad \text{for } k = (\text{greatest integer} \leq t); \quad (15)$$

$$\chi_+(t) = (k - t - \tfrac{1}{2}) \quad \text{for } k = (\text{least integer} \geq t); \quad (16)$$

and $\tilde{\chi}(t)$ given by (6).

Here we take $f(t) = E(t)$. Equation (14) then holds modulo errors $O(1)$, while (11) merely holds modulo $O(\lambda)$. Using (12) and (14) in place of the crude approximations (8), (11) in the derivation of (4), we arrive at the sharp formula (7) for $\text{sneg}(H)$. This concludes our introductory remarks on Theorem 1.

To understand atoms, we need to deal with potentials having a Coulomb singularity

$$V(x) \approx \frac{l(l+1)}{x^2} - \frac{Z}{x} + E^0 \quad \text{near } x=0. \quad (17)$$

In our application [FS5, FS5], we have $Z \gg 1$, $E^0 \sim Z^{4/3}$, and $l = O(Z^{1/3})$. When $l \ll Z^{1/3}$, the potential (17) is too singular to allow us to apply Theorem 1. Hence we need a more general result. In [FS2, FS3] we studied potentials $V(x)$ satisfying estimates

$$\left| \left(\frac{d}{dx} \right)^\alpha V(x) \right| \leq C_\alpha S(x) (B(x))^{-\alpha} \quad \text{when } x \in I, \quad (18)$$

for suitable weight functions $S(x)$, $B(x)$. Here I is an interval containing $\{x \in I_{\text{BVP}} : V(x) < 0\}$. Hypothesis (18) lets us treat all the potentials we need, simply by picking the proper weight functions $S(x)$, $B(x)$ and interval I . For instance, when V is given by (2), (3), then (18) holds with $S(x) = \lambda^2$, $B(x) = 1$, $I = I_{\text{BVP}}$. If V is given instead by (17), then we take $I \subset [l(l+1)/2Z, \infty)$ and pick $S(x) = Z/x$, $B(x) = x$ in the region where (17) holds. Here again, V satisfies (18).

The first main result of this paper is a version of Theorem 1 for general potentials satisfying (18). In the setting of (18), the natural analogue of hypothesis (3) is as follows.

$$V(x) \text{ takes its minimum at an interior point } x_0 \in I, \text{ and } V(x_0) < -cS(x_0). \text{ For } |x - x_0| \leq c_1 B(x_0) \text{ we have } V''(x) > cS(x)(B(x))^{-2}; \text{ while for } |x - x_0| \geq c_1 B(x_0) \text{ we have } |V'(x)| > cS(x)(B(x))^{-1}. \quad (19)$$

The number that plays the rôle of λ in Theorem 1 is

$$A = \left(\int_{V(x) < 0} \frac{dx}{(S(x))^{1/2} (B(x))^2} \right)^{-1}, \quad (20)$$

as explained in [FS2]. Now we can state our first main result, modulo technicalities.

FIRST WKB EIGENVALUE SUM THEOREM. *Let V satisfy (18), (19), and various technical conditions; and suppose A is large. Then $\text{sneg}(H)$ is given by (7), with $|\text{Error}| < C_e A^{e-2} |V(x_0)|$.*

For a precise statement of the result, we refer the reader to the relevant section of this paper.

When V satisfies (17) with relatively small l , we can make a crucial improvement over the preceding theorem, by comparing the eigenvalue sum for V with that of the Coulomb potential

$$V_c(x) = \frac{l(l+1)}{x^2} - \frac{Z}{x} + E^0. \quad (21)$$

We set $H_c = -d^2/dx^2 + V_c(x)$ on $(0, \infty)$. Our second main result is as follows.

SECOND WKB EIGENVALUE SUM THEOREM. *Let V satisfy (18), (19), and various technical conditions; and suppose Λ is large. Assume also that*

$$V(x) = V_c(x) \quad \text{for all } x \in (0, x_*], \quad (22)$$

with $x_0 \ll x_* \ll Z^{-1/3}$. Then

$$\begin{aligned} \text{sneg}(H) - \text{sneg}(H_c) = & -\frac{2}{3\pi} \int_0^\infty (-V(x))_+^{3/2} dx \\ & + \frac{1}{24\pi} \int_0^\infty V''(x)(-V(x))_+^{-1/2} dx \\ & + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) \\ & + \frac{2}{3\pi} \int_0^\infty (-V_c(x))_+^{3/2} dx \\ & - \frac{1}{24\pi} \int_0^\infty V_c''(x)(-V_c(x))_+^{-1/2} dx \\ & - \frac{2(E^0)^{3/2}}{Z} \tilde{\chi} \left(\frac{Z}{2(E^0)^{1/2}} - \sqrt{l(l+1)} - \frac{1}{2} \right) + \text{Error}, \end{aligned} \quad (23)$$

with

$$|\text{Error}| < C_\epsilon \Lambda^{\epsilon-2} \left(\frac{Z}{x_*} \right).$$

Again we refer to the reader to the relevant section of this paper for a precise statement of the result.

This paper contains additional results on $\text{sneg}(H)$ under various degenerate hypotheses, but we omit them from the introduction. In [FS5] we will apply the results of this paper to radially symmetric three-dimensional Schrödinger operators.

REVIEW OF EARLIER RESULTS

Here we gather results from [FS2, FS3] which we will need in this paper.

A. The WKB Eigenvalue Theorem

Set-up. We are given the following: A potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP} ; two positive functions $S(x)$, $B(x)$ defined on a subinterval $I \subset I_{\text{BVP}}$; two real numbers $E_0 \leq E_\infty$; positive numbers $\varepsilon < \frac{1}{100}$, $K > 1$, and $N > K\varepsilon^{-10}$. We define $N' = \lceil \varepsilon N / 500 \rceil$ and $N'' = \frac{3}{2}\varepsilon N' - K - 33$.

Our goal is to understand the eigenvalues of the self-adjoint operator $H = -d^2/dx^2 + V(x)$ on $L^2(I_{\text{BVP}})$, with Dirichlet or Neumann conditions at the endpoints.

Hypotheses

Assumptions on $V(x)$, $S(x)$, $B(x)$ in I .

(Hyp0) If $x, y \in I$ and $|x - y| < cB(x)$, then $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$.

(Hyp1) For $x \in I$ and $\alpha \geq 0$ we have $|(d/dx)^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(Hyp2) The equation $V(x) = E_0$ has two solutions $x_{\text{left}} < x_{\text{rt}}$ in I , and they satisfy $\text{dist}(x_{\text{left}}, \partial I) > cB(x_{\text{left}})$, $\text{dist}(x_{\text{rt}}, \partial I) > cB(x_{\text{rt}})$.

(Hyp3) For $x_{\text{left}} \leq x \leq x_{\text{left}} + c_1 B(x_{\text{left}})$ we have $-V'(x) > cS(x_{\text{left}}) B^{-1}(x_{\text{left}})$, and for $x_{\text{rt}} - c_1 B(x_{\text{rt}}) \leq x \leq x_{\text{rt}}$ we have $+V'(x) > cS(x_{\text{rt}}) B^{-1}(x_{\text{rt}})$.

(Hyp4) For $x_{\text{left}} + c_1 B(x_{\text{left}}) \leq x \leq x_{\text{rt}} - c_1 B(x_{\text{rt}})$ we have $cS(x) < E_0 - V(x) < CS(x)$.

To state the remaining hypotheses, we establish some notation. Set $\lambda(x) = S^{1/2}(x) B(x)$ for $x \in I$. Then set

$$\begin{aligned} B_{\text{left}} &= B(x_{\text{left}}), & S_{\text{left}} &= S(x_{\text{left}}), & \lambda_{\text{left}} &= \lambda(x_{\text{left}}). \\ B_{\text{rt}} &= B(x_{\text{rt}}), & S_{\text{rt}} &= S(x_{\text{rt}}), & \lambda_{\text{rt}} &= \lambda(x_{\text{rt}}). \end{aligned}$$

For $|E - E_0| < cS_{\text{left}}$, let $x_{\text{left}}(E)$ be the solution of $V(x) = E$ nearest x_{left} , and for $|E - E_0| < cS_{\text{rt}}$, let $x_{\text{rt}}(E)$ be the solution of $V(x) = E$ nearest x_{rt} . Define $S_{\min} = \inf_{x_{\text{left}} < x < x_{\text{rt}}} S(x)$ and $A = (\int_{x_{\text{left}}}^{x_{\text{rt}}} (dx/S^{1/2}(x) B^2(x)))$.

Our remaining hypotheses are as follows.

Assumptions on $V(x)$ in all of I_{BVP} .

(Hyp5) If $|E - E_0| < c_2 S_{\min}$ and $E \leq E_\infty$, then $V(x) > E$ for all $x \in I_{\text{BVP}} \setminus [x_{\text{left}}(E), x_{\text{rt}}(E)]$.

(Hyp6) If $x \in I_{\text{BVP}}$ satisfies $x < x_{\text{left}} - \frac{1}{2}\lambda_{\text{left}}^K B_{\text{left}}$ then $V(x) \geq E_\infty + 100/|x - x_{\text{left}}|^2$, and if $x \in I_{\text{BVP}}$ satisfies $x > x_{\text{rt}} + \frac{1}{2}\lambda_{\text{rt}}^K B_{\text{rt}}$, then $V(x) \geq E_\infty + 100/|x - x_{\text{rt}}|^2$.

Technical assumptions

$$(\text{Hyp7}) \quad \max_{x \in I} S(x) \leq \lambda_{\text{left}}^K S_{\text{left}} \quad \text{and} \quad \max_{x \in I} S(x) \leq \lambda_{\text{rt}}^K S_{\text{rt}}.$$

$$(\text{Hyp8}) \quad \int_{x_{\text{left}}}^{x_{\text{rt}}} (dx/S^{1/2}(x)) \leq A^K \cdot \min(S_{\text{left}}^{-1/2} B_{\text{left}}, S_{\text{rt}}^{-1/2} B_{\text{rt}}).$$

$$(\text{Hyp9}) \quad \left[\int_{x_{\text{left}}}^{x_{\text{rt}}} (dx/S^{1/2}(x) B^4(x)) \right] \cdot \left[\int_{x_{\text{left}}}^{x_{\text{rt}}} (dx/S^{1/2}(x)) \right] \leq A^K.$$

WKB Condition.

(Hyp10) A is bounded below by a positive constant depending only on ε, K, N , and on c, C, c_1, c_2, C_α in (Hyp0)–(Hyp4).

Definitions and Basic Properties of Phases

Assume hypotheses (Hyp0)–(Hyp10). For $|E - E_0| < c_\# S_{\min}$, define

$$\phi(E) = \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(x))^{1/2} dx$$

and

$$\psi(E) = \lim_{\delta \rightarrow 0+} \left[\int_{x_{\text{left}}(E) + \delta}^{x_{\text{rt}}(E) - \delta} V''(x)(E - V(x))^{-3/2} dx - q(E) \delta^{-1/2} \right]$$

with $q(E)$ uniquely specified by demanding the finiteness of the limit.

LEMMA A1. For $|E - E_0| < c_\# S_{\min}$ we have

$$\left| \left(\frac{d}{dE} \right)^\beta \phi(E) \right| \leq C_\#^\beta \int_{x_{\text{left}}}^{x_{\text{rt}}} S^{1/2-\beta}(x) dx$$

and

$$\left| \left(\frac{d}{dE} \right)^\beta \psi(E) \right| \leq C_\#^\beta \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2+\beta}(x) B^2(x)}.$$

Also, $c_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} (dx/S^{1/2}(x)) < d\phi(E)/dE < C_\# \int_{x_{\text{left}}}^{x_{\text{rt}}} (dx/S^{1/2}(x))$.

The constants $c_\#, C_\#, C_\#^\beta$ depend only on $\varepsilon, K, N, c, C, c_1, c_2, C_\alpha$ in the hypotheses (Hyp0)–(Hyp4).

WKB EIGENVALUE THEOREM. If (Hyp0)–(Hyp10) hold, then there is a function $\Phi(E)$ on $[E_0 - c_\# S_{\min}, E_0 + c_\# S_{\min}]$ and there are numbers $E_{k_{\min}}, E_{k_{\min}+1}, \dots, E_{k_{\max}} \leq E_\infty$ with the following properties.

$$(A) \quad \Phi(E) = \pm \pi/2 + \phi(E) + \frac{1}{48} \psi(E) + \phi_{\text{error}}(E), \text{ with}$$

$$\left| \left(\frac{d}{dE} \right)^\beta \phi_{\text{error}} \right| \leq C_\#^\beta A^{-1} \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{S^{1/2+\beta}(x) B^2(x)}, \quad \text{all } \beta \geq 0.$$

(B) $\{k_{\min}, k_{\min} + 1, \dots, k_{\max}\}$ is exactly the set of integers k for which $|\Phi(E) - \pi k| < C_{\#} A^{-N''}$ for some $E \in [E_0 - \frac{1}{4}c_{\#} S_{\min}, E_0 + \frac{1}{4}c_{\#} S_{\min}] \cap (-\infty, E_{\infty}]$.

(C) If $k_{\min} \leq k < k_{\max}$, then E_k is an eigenvalue of H .

(D) If $k = k_{\max}$, then E_k is either an eigenvalue of H or equal to E_{∞} .

(E) For $k_{\min} \leq k \leq k_{\max}$ we have $|E_k - E_0| < c_{\#} S_{\min}$ and $|\Phi(E_k) - \pi k| < C_{\#} A^{-N''}$.

(F) Every eigenvalue of H in the interval $[E_0 - \frac{1}{4}c_{\#} S_{\min}, E_0 + \frac{1}{4}c_{\#} S_{\min}] \cap (-\infty, E_{\infty}]$ is one of the E_k ($k_{\min} \leq k \leq k_{\max}$).

The constants $c_{\#}$, $C_{\#}^{\beta}$ depend only on $\varepsilon, K, N, c, C, c_1, c_2, C_{\alpha}$ in (Hyp0)–(Hyp5).

Remark. Perhaps $E_{k_{\min}}$ or $E_{k_{\max}}$ or both lie slightly outside

$$[E_0 - \frac{1}{4}c_{\#} S_{\min}, E_0 + \frac{1}{4}c_{\#} S_{\min}].$$

Note that the choice of sign in (A) affects only the indexing of the eigenvalues, not the content of the theorem.

B. The WKB Theorem on Low Eigenvalues

Let $\varepsilon, K, N > 0$ be given, with $\varepsilon N \geq 100$. Let $V(x)$ be a potential defined on a (possibly unbounded) interval I_{BVP} . Let S, B be positive numbers, and let $x_0 \in I_{\text{BVP}}$ be given. Define $\lambda = S^{1/2}B$. Let E_{∞} be a given energy, with $E_{\infty} > V(x_0)$. We make the following assumptions.

(H0*) $I = \{|x - x_0| < cB\} \subset I_{\text{BVP}}$.

(H1*) $|(d/dx)^{\alpha} V(x)| \leq C_{\alpha} S B^{-\alpha}$ in I .

(H2*) $(d^2/dx^2)V \geq c'SB^{-2}$ in I .

(H3*) $V'(x_0) = 0$.

(H4*) For $x \in I_{\text{BVP}} \setminus I$ we have $V(x) \geq \min\{E_{\infty}, V(x_0) + c''\lambda^{-2\varepsilon}S\}$.

(H5*) For $x \in I_{\text{BVP}}$ with $|x - x_0| > \frac{1}{2}\lambda^K B$, we have $V(x) \geq E_{\infty} + 1000/|x - x_0|^2$.

(H6*) λ is bounded below by a positive constant depending only on c, c', c'', C_{α} in (H0*)–(H4*), and on ε, K, N .

Let $H = -d^2/dx^2 + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet or Neumann conditions at the endpoints.

For $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon}S$, define $x_{\text{left}}(E) < x_{\text{rt}}(E)$ to be the two values of $x \in I$ at which $V(x) = E$. Then define

$$\begin{aligned}\phi(E) &= \int_{x_{\text{left}}(E)}^{x_{\text{rt}}(E)} (E - V(x))^{1/2} dx \\ \psi(E) &= \lim_{\delta \rightarrow 0+} \left[\int_{\substack{E - V(x) > \delta \\ x \in I}} V''(x)(E - V(x))^{-3/2} dx - q(E) \delta^{-1/2} \right] \\ &= \lim_{\delta_{\text{left}}, \delta_{\text{rt}} \rightarrow 0+} \left[\int_{x_{\text{left}}(E) + \delta_{\text{left}}}^{x_{\text{rt}}(E) - \delta_{\text{rt}}} V''(x)(E - V(x))^{-3/2} dx - q_{\text{left}}(E) \delta_{\text{left}}^{-1/2} \right. \\ &\quad \left. - q_{\text{rt}}(E) \delta_{\text{rt}}^{-1/2} \right]\end{aligned}$$

with $q(E)$, $q_{\text{left}}(E)$, $q_{\text{rt}}(E)$ uniquely specified by demanding the finiteness of the limits.

LEMMA B1. *The phases $\phi(E)$, $\psi(E)$ satisfy the estimates*

$$\begin{aligned}\left| \left(\frac{d}{dE} \right)^\beta \phi(E) \right| &\leq C_{\#}^\beta \lambda S^{-\beta} \\ \left| \left(\frac{d}{dE} \right)^\beta \psi(E) \right| &\leq C_{\#}^\beta \lambda^{-1} S^{-\beta} \\ \frac{d}{dE} \phi(E) &\geq c_{\#} \lambda S^{-1}\end{aligned}$$

for $V(x_0) < E < V(x_0) + \lambda^{-2\varepsilon}S$.

The constants $c_{\#}$, $C_{\#}^\beta$ depend only on c , c' , c'' , C_α , ε , K , N in hypotheses (H0*)–(H6*).

WKB THEOREM ON LOW EIGENVALUES. Assume (H0*)–(H6*). Then there is a finite sequence $E_0, E_1, \dots, E_{k_{\text{max}}}$ with the following properties.

(a) Let $w = \phi(E_*) + \frac{1}{48}\psi(E_*)$ with $E_* = \min\{E_\infty, V(x_0) + c_{\#} \lambda^{-2\varepsilon}S\}$, and let \bar{n} be the largest integer with $\pi(\bar{n} + \frac{1}{2}) \leq w$. If $\min_{k \in \mathbb{Z}} |w - \pi(k + \frac{1}{2})| > C_{\#} \lambda^{-2+4\varepsilon}$, then $k_{\text{max}} = \bar{n}$. In any case, $|k_{\text{max}} - \bar{n}| \leq 1$.

(b) If $0 \leq k < k_{\text{max}}$, then E_k is an eigenvalue of H .

(c) Either $E_{k_{\text{max}}} = E_\infty$ or else $E_{k_{\text{max}}}$ is an eigenvalue of H .

(d) Every eigenvalue E of H satisfying $E \leq E_\infty$, $|E - V(x_0)| < c_{\#} \lambda^{-2\varepsilon}S$ is one of the E_k .

(e) For $0 \leq k \leq k_{\text{max}}$ we have $V(x_0) < E_k < V(x_0) + 2c_{\#} \lambda^{-2\varepsilon}S$ and $|\phi(E_k) + \frac{1}{48}\psi(E_k) - \pi(k + \frac{1}{2})| \leq C_{\#} \lambda^{-2+4\varepsilon}$.

The constants c_* , C_* depend only on $\varepsilon, K, N, c, c', c'', C_\alpha$ in hypotheses (H0*)–(H6*).

C. The Basic Hypotheses for Potentials

Set-up. We are given positive numbers $\varepsilon, K, N, \hat{c}$; two intervals $I \subset I_{\text{BVP}}$ (possibly unbounded); a point $x_0 \in I$; a potential $V(x)$ defined on I_{BVP} ; and two positive functions $S(x), B(x)$ defined on I . Our assumptions are as follows.

Assumptions Concerning $V(x), S(x), B(x)$ on I .

(Z0) If $x, y \in I$ and $|x - y| < cB(x)$, then $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$.

(Z1) If $x \in I$ and $\alpha \geq 0$, then $|(d/dx)^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(Z2) The set $\{x \in I \mid V(x) < 0\}$ is a non-empty interval $(x_{\text{left}}, x_{\text{rt}})$, with $\text{dist}(x_{\text{left}}, \partial I) > cB(x_{\text{left}})$ and $\text{dist}(x_{\text{rt}}, \partial I) > cB(x_{\text{rt}})$.

(Z3) We have $V(x_0) < -cS(x_0)$, $V'(x_0) = 0$; and for $|x - x_0| \leq c_1 B(x_0)$ we have $V''(x) \geq cS(x_0) B^{-2}(x_0)$.

(Z4) For $x_{\text{left}} \leq x \leq x_0 - c_1 B(x_0)$ we have $-V'(x) > cS(x) B^{-1}(x)$; and for $x_0 + c_1 B(x_0) \leq x \leq x_{\text{rt}}$ we have $+V'(x) > cS(x) B^{-1}(x)$.

Define $\lambda(x) = S^{1/2}(x) B(x)$ for $x \in I$, and set

$$A = \left(\int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{\lambda(x) B(x)} \right)^{-1}.$$

Assumptions Concerning $V(x)$ on all of I_{BVP} .

(Z5) We have $V(x) > 0$ for all $x \in I_{\text{BVP}} \setminus [x_{\text{left}}, x_{\text{rt}}]$.

(Z6) For all $x \in I_{\text{BVP}}$ with $x < x_{\text{left}} - A^K B(x_{\text{left}})$, we have $V(x) \geq 1000/|x - x_{\text{left}}|^2$; and for all $x \in I_{\text{BVP}}$ with $x > x_{\text{rt}} + A^K B(x_{\text{rt}})$, we have $V(x) \geq 1000/|x - x_{\text{rt}}|^2$.

Polynomial Growth Assumptions on $S(x), B(x), I$.

(Z7) We have $\max_{x \in I} B(x) < A^K \min_{x \in I} B(x)$; $\max_{x \in I} S(x) < A^K \min_{x \in I} S(x)$; and $|I| < A^K \cdot \min_{x \in I} B(x)$.

Smallness of the Constant \hat{c} .

(Z8) The constant \hat{c} is bounded above by a certain small, positive number determined by $\varepsilon, K, N, c, C, c_1, C_\alpha$.

The WKB Hypothesis.

(Z9) A is bounded below by a certain large, positive number determined by $\varepsilon, K, N, c, C, c_1, \hat{c}, C_\alpha$.

Let E_k and $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions of $-d^2/dx^2 + V(x)$ on I_{BVP} , with Dirichlet or Neumann boundary conditions. In [FS3] we studied the density $\rho(x) = \sum_{E_k \leq 0} |u_k(x)|^2$ under the assumptions (Z0)–(Z9). Our result was called the “WKB Density Theorem.” This paper will study $\text{sneg}(H) = \sum_{E_k \leq 0} E_k$ under the same assumptions.

Remark. We have kept hypothesis (Z8) from [FS3], even though the constant \hat{c} plays no rôle in this paper, simply to keep the same hypothesis as before.

D. The WKB Eigenvalue Theorems for Potentials with Weak Turning Points

Set-up. We are given an energy E_0 and a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP} . The interval I_{BVP} is partitioned into subintervals $I_{\text{farleft}}, I_{\text{left}}, I_{\text{center}}, I_{\text{rt}}, I_{\text{farrt}}$ with I_{farleft} to the left of I_{left} , I_{left} to the left of I_{center} , etc. Here, I_{farleft} and I_{farright} may be empty. On I_{center} we are given positive weight functions $S(x)$, $B(x)$. Set $\lambda(x) = S^{1/2}(x) B(x)$ and $A = (\int_{I_{\text{center}}} (dx/\lambda(x) B(x)))^{-1}$. We make the following assumptions.

Hypotheses

(H0) I_{center} is non-empty, and for $x, y \in I_{\text{center}}$ with $|x - y| < cB(x)$ we have $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$, and $|I_{\text{center}}| > cB(x)$.

(H1) For $x \in I_{\text{center}}$ we have $|(d/dx)^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$ and $cS(x) < E_0 - V(x) < CS(x)$.

(H2) A is bounded below by a large number depending only on c, C, C_α in (H0), (H1).

(H3) $I_{\text{left}}, I_{\text{rt}}$ are non-empty. If $I_{\text{center}} = [x_{\text{left}}, x_{\text{rt}}]$, then we have $|I_{\text{left}}| \leq \underline{C}B(x_{\text{left}})$, $|I_{\text{rt}}| \leq \underline{C}B(x_{\text{rt}})$, $\lambda(x_{\text{left}}) \leq \underline{C}$, $\lambda(x_{\text{rt}}) \leq \underline{C}$.

(H4) If $I_{\text{left}} = [x_{\text{farleft}}, x_{\text{left}}]$, then we have $|V(x) - E_0| \leq \underline{C} |I_{\text{left}}|^{-1} (x - \hat{x}_{\text{farleft}})^{-1}$ in I_{left} . Here $\hat{x}_{\text{farleft}} \leq x_{\text{farleft}}$ with strict inequality unless $I_{\text{farleft}} = \emptyset$.

(H5) For $x \in I_{\text{rt}}$ we have $|V(x) - E_0| \leq \underline{C} |I_{\text{rt}}|^{-2}$.

(H6) For $x \in I_{\text{farleft}}$ we have $V(x) - E_0 \geq \underline{c} |I_{\text{left}}|^{-2}$. $V(x)$ is C^∞ in the interior of I_{farleft} .

(H7) For $x \in I_{\text{farrt}}$ we have $V(x) - E_0 \geq -10^{-9}/(x - x_{\text{rt}})^2$.

Our goal here is to understand the eigenvalues of $H = -d^2/dx^2 + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet boundary conditions.

Denote by C_* a constant depending on c, C, C_α in (H0), (H1). Denote by C_*, c_* , etc., constants depending only on $c, C, C_\alpha, \underline{c}, \underline{C}$ in (H0)–(H7).

Note that I_{left} and I_{rt} do not play completely analogous rôles in our hypotheses.

THEOREM D1. Under the assumptions (H0)–(H7) we have

$$|(\text{Number of eigenvalues of } H < E_0) - \frac{1}{\pi} \int_{I_{\text{center}}} (E_0 - V(t))^{1/2} dt| \leq C_*.$$

As an application of this theorem, we study the following situation.

Set-up. $V(x)$ is a potential defined on a (possibly unbounded) interval I_{BVP} . We are given a subinterval $I \subset I_{\text{BVP}}$ and weight functions $S(x)$, $B(x) > 0$ defined on I . Set $\lambda(x) = S^{1/2}(x) B(x)$. We are given an energy E_0 .

We make the following assumptions.

Hypotheses

(H0) For $x, y \in I$ with $|x - y| < cB(x)$ we have $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$, and $|I| > cB(x)$.

(H1) For $x \in I$ we have $|(d/dx)^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(H2) $\{x \in I_{\text{BVP}} \mid V(x) < E_0\} = (x_{\text{left}}, x_{\text{right}}) \subset I$ with $\text{dist}(x_{\text{left}}, \partial I) > cB(x_{\text{left}})$, $\text{dist}(x_{\text{right}}, \partial I) > cB(x_{\text{right}})$.

(H3) In $[x_{\text{left}}, x_{\text{left}} + c_1 B(x_{\text{left}})]$ we have $-V'(x) \geq cS(x_{\text{left}})/B(x_{\text{left}})$, and in $[x_{\text{right}} - c_1 B(x_{\text{right}}), x_{\text{right}}]$ we have $+V'(x) \geq cS(x_{\text{right}})/B(x_{\text{right}})$.

(H4) In $[x_{\text{left}} + c_1 B(x_{\text{left}}), x_{\text{right}} - c_1 B(x_{\text{right}})]$ we have $cS(x) < E_0 - V(x) < CS(x)$.

(H5) In $I_{\text{BVP}}^{\text{interior}} \cap (-\infty, x_{\text{left}})$, $V(x)$ is decreasing and C^∞ .

(H6) $A = (\int_{x_{\text{left}}}^{x_{\text{right}}} (dx/\lambda(x) B(x)))^{-1}$ is greater than a large positive number depending only on c, C, c_1, C_α in (H0)–(H4).

Let $H = -d^2/dx^2 + V(x)$ on $L^2(I_{\text{BVP}})$ with Dirichlet boundary conditions.

THEOREM D2. Under assumptions (H0)–(H6) above, we have

$$|(\text{Number of eigenvalues of } H < E_0) - \frac{1}{\pi} \int_{I_{\text{BVP}}} (E_0 - V(t))_+^{1/2} dt| \leq C_*.$$

The constant C_* depends only on c, C, c_1, C_α in (H0)–(H4).

E. Properties of Degenerate Potentials, I

From the section The Density for Degenerate One-Dimensional Potentials III in [FS3], we recall the following results.

We are given a potential $V(x)$, smooth on $(0, \infty)$. We take $B(x) = x$ and let $S(x)$ be a positive function on $I = [x_0, x_1] \subset (0, \infty)$. As usual, we set $\lambda(x) = S^{1/2}(x) B(x)$ on I . In addition to x_0, x_1 , we are given other points $x_{\text{small}}, x_{\text{big}}, x_{\text{crit}}, x_* \in (0, \infty)$, with

$$0 < x_{\text{small}} < \frac{1}{2}x_0, \quad 2x_0 < x_{\text{crit}} < \frac{1}{2}x_*, \quad x_* < \frac{1}{16}x_1, \quad 2x_1 < x_{\text{big}}. \quad (1)$$

Set $H = -d^2/dx^2 + V(x)$ on $(0, \infty)$ with Dirichlet boundary conditions. Let E_k , $u_k(x)$ be the eigenvalues and (normalized) eigenfunctions of H .

In addition to (1), we make the following assumptions.

Hypotheses

- (Z0) If $x, y \in I$ and $|x - y| < \frac{1}{2}B(x)$, then $c < S(y)/S(x) < C$.
- (Z1) If $x \in I$, then $|(d/dx)^2 V(x)| \leq C_x S(x) B^{-\alpha}(x)$.
- (Z2) If $x \in I$, then $V(x) < -cS(x)$ and $V'(x) > cS(x) B^{-1}(x)$.
- (Z3) $\Lambda = (\int_I (dx/\lambda(x) B(x)))^{-1}$ is greater than a certain large, positive number determined by c, C, C_x in (Z0)–(Z2).
- (Z4) For $x \in (0, x_{\text{small}}]$ we have $V(x) \geq \underline{c}x_0^{-2}$.
- (Z5) For $x \in [x_{\text{small}}, x_0]$ we have $|V(x)| \leq \underline{C}x_0^{-2}$.
- (Z6) We have $x_{\text{big}} < \underline{C}x_1$ and $V(x)$ is increasing in $[x_1, x_{\text{big}}]$.
- (Z7) For $x \in [x_1/8, x_{\text{big}}]$, we have $|V(x)| \leq \underline{C}x_1^{-2}$.
- (Z8) For $x \in [x_{\text{big}}, \infty)$, we have $V(x) \geq 0$.
- (Z9) For $E \in [V(x_*), 0]$ we have

$$\int_{x_0}^{x_{\text{crit}}} (E - V(x))^{-1/2} dx \leq \delta \cdot \int_{x_0}^{x_*/2} (E - V(x))^{-1/2} dx.$$

We denote by $c_{\#}, C_{\#}$, etc., constants that depend only on c, C, C_x ; while c_*, C_* , etc., denote constants that depend also on $\underline{c}, \underline{C}$.

LEMMA E1. Set $E_0 = 0$, $I_{\text{center}} = [x_0, x_1]$, $I_{\text{left}} = [x_{\text{small}}, x_0]$, $I_{\text{rt}} = [x_1, x_{\text{big}}]$, $I_{\text{farleft}} = (0, x_{\text{small}}]$, $I_{\text{farrt}} = [x_{\text{big}}, \infty)$. Then the hypotheses (H0)–(H7) of Theorem D1 in the section The Eigenvalue Theorems for Potentials with Weak Turning Points are satisfied. The constants called c, C, C_x in (H0)–(H7) may be taken to be of the form $C_{\#}$. The constants called $\underline{c}, \underline{C}$ in (H0)–(H7) may be taken to be of the form C_* .

LEMMA E2. Suppose $2x_0 < \tilde{x} < \frac{1}{2}x_1$. Set $\tilde{E} = V(\tilde{x})$, and define: $\tilde{V}(x) = V(x) - \tilde{E}$, $\tilde{E}_0 = 0$, $\tilde{I}_{\text{center}} = [x_0, \tilde{x} - \hat{C}_{\#} \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$ with $\hat{C}_{\#}$ picked large enough, $\tilde{I}_{\text{left}} = [x_{\text{small}}, x_0]$, $\tilde{I}_{\text{farleft}} = (0, x_{\text{small}}]$, $\tilde{I}_{\text{rt}} = [\tilde{x} - \hat{C}_{\#} \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x} + \hat{C}_{\#} \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$, $\tilde{I}_{\text{farrt}} = [\tilde{x} + \hat{C}_{\#} \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \infty)$. On $\tilde{I}_{\text{center}}$, define $\tilde{B}(x) = \min(x, \tilde{x} - x)$, $\tilde{S}(x) = S(x) \cdot ((\tilde{x} - x)/\tilde{x})$. Then $\tilde{V}(x)$, \tilde{E}_0 , $\tilde{I}_{\text{farleft}} \cdots \tilde{I}_{\text{farrt}}$, $\tilde{S}(x)$, $\tilde{B}(\tilde{x})$ satisfy hypotheses (H0)–(H7) in the section on WKB theory with weak turning points. The constants called c, C, C_x in (H0)–(H7) may be taken to have the form $C_{\#}$. The constants called $\underline{c}, \underline{C}$ in (H0)–(H7) may be taken to have the form C_* .

F. *Properties of Degenerate Potentials, II*

From the section *The Density for Degenerate One-Dimensional Potentials IV* in [FS3], we recall the following results.

We are given a smooth potential $V(x)$ on $(0, \infty)$. We take $B(x) = x$, and let $S(x)$ be a positive function on $I = [x_0, x_1] \subset (0, \infty)$. Let $\lambda(x) = S^{1/2}(x) B(x)$ as usual. We are given $x_{\text{crit}}, x_*, x_{\text{big}}$, satisfying

$$16x_0 < x_{\text{crit}}, \quad 16x_{\text{crit}} < \frac{1}{10}x_*, \quad \frac{16}{10}x_* < x_1, \quad 16x_1 < x_{\text{big}}. \quad (1)$$

Set $H = -d^2/dx^2 + V(x)$ on $(0, \infty)$, with Dirichlet boundary conditions. (We have changed notation slightly from [FS3]; our present x_* differs from that in [FS3] by a factor of 10.) In addition to (1), we make the following assumptions.

Hypotheses

(Z0⁺) If $x, y \in I$ and $|x - y| < \frac{1}{2}B(x)$, then $c < S(y)/S(x) < C$.

(Z1⁺) If $x \in I$ and $\alpha \geq 0$, then $|(d/dx)^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(Z2⁺) If $x \in I$, then $V(x) < -cS(x)$ and $V'(x) > cS(x) B^{-1}(x)$.

(Z3⁺) $A = (\int_I (dx/\lambda(x) B(x)))^{-1}$ is greater than a certain large, positive number determined by c, C, C_α in (Z0⁺)–(Z2⁺).

(Z4⁺) $|V(x)| \leq \underline{C}/(x_0 x)$ for $x \in (0, x_0]$.

(Z5⁺) $V(x)$ is increasing and negative in $[x_1/8, x_{\text{big}}]$ and satisfies there $|V(x)| < \underline{C}x_1^{-2}$. Also, $x_{\text{big}} < \underline{C}x_1$.

(Z6⁺) $V(x) \geq -10^{-9}x^{-2}$ for $x \in [x_{\text{big}}, \infty)$.

(Z7⁺) For $E \in [V(x_*/10), 0]$, we have

$$\int_{x_0}^{x_{\text{crit}}} (E - V(x))^{-1/2} dx \leq \delta \cdot \int_{x_0}^{x^*/20} (E - V(x))^{-1/2} dx.$$

We denote by c_*, C_* , etc., constants that depend only on c, C, C_α in (Z0⁺)–(Z7⁺); while c_*, C_* , etc., denote constants that depend also on \underline{C} .

LEMMA F1. Set $I_{\text{farleft}} = \emptyset$, $I_{\text{left}} = (0, x_0]$, $I_{\text{center}} = [x_0, x_1]$, $I_{\text{rt}} = [x_1, x_{\text{big}}]$, $I_{\text{farrrt}} = [x_{\text{big}}, \infty)$, $E_0 = 0$. Then the hypotheses (H0)–(H7), from the section on *WKB with weak turning points*, are satisfied. The constants called c, C, C_α in (H0)–(H7) may be taken of the form C_* . The constants called $\underline{c}, \underline{C}$ in (H0)–(H7) may be taken of the form C_* .

LEMMA F2. Suppose $\tilde{E} = V(\tilde{x})$ with $\tilde{x} \in [\frac{1}{10}x_*, \frac{1}{4}x_1]$. Take $\tilde{V}(x) = V(x) - \tilde{E}$, $\tilde{S}(x) = S(x) \cdot ((\tilde{x} - x)/\tilde{x})$, $\tilde{B}(x) = \min(x, \tilde{x} - x)$, $\tilde{E}_0 = 0$, $\tilde{I}_{\text{farleft}} = \emptyset$, $\tilde{I}_{\text{left}} = (0, x_0]$, $\tilde{I}_{\text{center}} = [x_0, \tilde{x} - \hat{C}_* \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$ with \hat{C}_* picked large

enough, $\tilde{I}_{\text{rt}} = [\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}]$, $\tilde{I}_{\text{farri}} = [\tilde{x} + \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \infty)$. Then the hypotheses (H $\hat{0}$)–(H $\hat{7}$), in the section on WKB with weak turning points, are satisfied. The constants called c, C, C_α in (H $\hat{0}$)–(H $\hat{7}$) may be taken of the form $C_\#$. The constants called $\underline{c}, \underline{C}$ in (H $\hat{0}$)–(H $\hat{7}$) may be taken of the form C_\star .

G. Properties of Degenerate Potentials, III

In the section The Density for Degenerate One-Dimensional Potentials II in [FS3], we studied the following class of degenerate potentials.

Set-up. We are given a potential $V(x)$ defined on a (possibly unbounded) interval I_{BVP} ; positive functions $S(x)$, $B(x)$, defined on a subinterval $I \subset I_{\text{BVP}}$; a point $x_{\text{crit}} \in I_{\text{BVP}}$; an energy $E_{\text{crit}} \leq 0$; and a number δ strictly between 0 and 1.

Assumptions.

(Z $\bar{0}$) For $x, y \in I$ with $|x - y| < cB(x)$, we have $c < B(y)/B(x) < C$ and $c < S(y)/S(x) < C$, and $|I| > cB(x)$.

(Z $\bar{1}$) For $x \in I$ and $\alpha \geq 0$ we have $|(d/dx)^\alpha V(x)| \leq C_\alpha S(x) B^{-\alpha}(x)$.

(Z $\bar{2}$) For $E_{\text{crit}} \leq E \leq 0$, the set $\{x \in I_{\text{BVP}} \mid V(x) \leq E\}$ is a non-empty interval $(x_{\text{left}}(E), x_{\text{rt}}(E))$ contained in I , with $\text{dist}(x_{\text{left}}(E), \partial I) > cB(x_{\text{left}}(E))$ and $\text{dist}(x_{\text{rt}}(E), \partial I) > cB(x_{\text{rt}}(E))$.

(Z $\bar{3}$) For $E_{\text{crit}} \leq E \leq 0$, we have $-V'(x) \geq cS(x) B^{-1}(x)$ for $x \in [x_{\text{left}}(E), x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E))]$ and $+V'(x) \geq cS(x) B^{-1}(x)$ for $x \in [x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E)), x_{\text{rt}}(E)]$.

(Z $\bar{4}$) For $E_{\text{crit}} \leq E \leq 0$, we have $cS(x) < E - V(x) < CS(x)$ for $x \in [x_{\text{left}}(E) + c_1 B(x_{\text{left}}(E)), x_{\text{rt}}(E) - c_1 B(x_{\text{rt}}(E))]$.

(Z $\bar{5}$) $V(x)$ is decreasing and C^∞ on $I_{\text{BVP}}^{\text{interior}} \cap (-\infty, x_{\text{left}}(0)]$.

(Z $\bar{6}$) For $E_{\text{crit}} \leq E \leq 0$, we have $x_{\text{left}}(E) + cB(x_{\text{left}}(E)) \leq x_{\text{crit}}$.

(Z $\bar{7}$) For $E_{\text{crit}} \leq E \leq 0$, we have

$$\int_{I_{\text{BVP}} \cap (-\infty, x_{\text{crit}}]} (E - V(t))_+^{-1/2} dt \leq \delta \int_{I_{\text{BVP}}} (E - V(t))_+^{-1/2} dt.$$

(Z $\bar{8}$) $\Lambda \equiv (\int_{x_{\text{left}}(0)}^{x_{\text{rt}}(0)} (dx/\lambda(x) B(x)))^{-1}$ is greater than a certain large, positive number determined by c, C, c_1, C_α above.

Here, $\lambda(x) = S^{1/2}(x) B(x)$ as usual.

Note that (Z $\bar{0}$)–(Z $\bar{8}$) imply the hypotheses (H $\bar{0}$)–(H $\bar{6}$) of Theorem D2 in the section on WKB theory with weak turning points, for any $E_0 \in [E_{\text{crit}}, 0]$.

H. Approximating Sums by Integrals

From [FS3] we recall the following result.

LEMMA ON RIEMANN SUMS. *Let $f(t)$, $\sigma(t)$, $\tau(t)$ be defined on a non-empty interval $[a, b]$. Suppose $\sigma(t) > 0$, $\tau(t) \geq 1$ in $[a, b]$; and assume that whenever $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < c\tau(t_1)$, we have $c < \tau(t_2)/\tau(t_1) < C$ and $c < \sigma(t_2)/\sigma(t_1) < C$. Finally, assume $|(d/dt)^m f(t)| \leq C_m \sigma(t) \tau^{-m}(t)$ for $t \in [a, b]$. Then $\sum_{k \in \mathbb{Z} \cap [a, b]} f(k) = \int_a^b f(t) dt - f(b) \chi_-(b) - f(a) \chi_+(a) + \frac{1}{2} f'(b) \tilde{\chi}(b) - \frac{1}{2} f'(a) \tilde{\chi}(a) + \text{Error}$ with $|\text{Error}| \leq C' \sigma(a) \tau^{-2}(a) + C' \sigma(b) \tau^{-2}(b) + C'_N \int_a^b \sigma(t) \tau^{-N}(t) dt$. Here, C' depends only on c, C, C_m ; and C'_N depends only on c, C, C_m, N . If $f(t) = 0$ to infinite order at $t = a$, then we have the sharper estimate $|\text{Error}| \leq C' \sigma(b) \tau^{-2}(b) + C'_N \int_a^b \sigma(t) \tau^{-N}(t) dt$, with C', C'_N as before. Similarly, if $f(t) = 0$ to infinite order at $t = b$, then $|\text{Error}| \leq C' \sigma(a) \tau^{-2}(a) + C'_N \int_a^b \sigma(t) \tau^{-N}(t) dt$. If $f(t) = 0$ to infinite order at both $t = a$ and $t = b$, then $|\text{Error}| \leq C'_N \int_a^b \sigma(t) \tau^{-N}(t) dt$.*

TRUNCATED EIGENVALUE SUMS

We adopt the notation and hypotheses (Z0)–(Z9) of the WKB Density Theorem, and let E_* be a given energy strictly between zero and the minimum of the potential. Our goal is to compute the sum of the eigenvalues in $[E_*, 0]$ modulo a small error. We assume that the phase $\phi(E_*)$ differs from $\pi \cdot (\text{integer})$ by at most Λ^{-2} . Thus, $\phi(E_*) + \frac{1}{48} \psi(E_*)$ is not close to $\pi(k + \frac{1}{2})$ for any integer k . We assume also $\phi(E_*) < \phi(0) - 1$.

We use c_*, C_* , etc., to denote constants that depend only on $\varepsilon, K, N, c, C, c_1, C_x, \hat{c}$ in the hypotheses (Z0)–(Z9) of the WKB Density Theorem.

Our first step is to use the WKB Eigenvalue Theorem and the WKB Theorem on Low Eigenvalues to identify the eigenvalues of $-(d^2/dx^2 + V(x))$. Later, we will use the Lemma on Riemann Sums to compute the sum of the eigenvalues. We start by checking the hypothesis of the WKB Eigenvalue Theorem.

LEMMA 1. *Suppose $V(x_0) + c_* S(x_0) \leq E_0 \leq 0$. Then the hypotheses (Hyp0)–(Hyp10) of the WKB theorems are satisfied, with $E_\infty = 0$ and with $300K$ in place of K . The constants called c, C, c_1, c_2, C_x in (Hyp0)–(Hyp10) all have the form C_* . The number called Λ in (Hyp0)–(Hyp10) is greater than or equal to our present Λ .*

Proof. This is just Lemma 1 from the section Combining Microlocalized Results in [FS3] with $\varphi(E) \equiv 0$ and $\hat{C}_\beta \equiv 0$. ■

LEMMA 2. Suppose $2^{-2(m+1)}c_{\#}S(x_0) < E_0 - V(x_0) \leq 2^{-2m}c_{\#}S(x_0)$, with $\lambda^{-3e}(x_0) \leq 2^{-m} \leq 1$. Set $S_m = 2^{-2m}S(x_0)$, $B_m = 2^{-m}B(x_0)$, $I_m = \{|x - x_0| < C_{\#}B_m\}$. Then we can find an energy $\tilde{E}_m \leq 0$ with $\tilde{E}_m - E_0 \sim 2^{-2m}S(x_0)$, $\min_{k \in \mathbb{Z}} |\phi(\tilde{E}_m) - \pi(k + \frac{1}{2})| \geq \frac{1}{20}$, and satisfying the following. The hypotheses (Hyp0)–(Hyp10) of the WKB theorems are satisfied, with $V(x) - \tilde{E}_m$ in place of $V(x)$; S_m and B_m in place of S and B ; I_m in place of I ; with $E_{\infty} = 0$; with $E_0 - \tilde{E}_m$ in place of E_0 ; and with $300K$ in place of K . The constants called c, C, c_1, c_2, C_x in (Hyp0)–(Hyp10) all have the form $C_{\#}$. The number called λ in (Hyp0)–(Hyp10) is $\sim 2^{-2m}\lambda(x_0)$.

Proof. For $V(x_0) < E < V(x_0) + c_{\#}S(x_0)$ we have $\phi'(E) \sim \lambda(x_0)S^{-1}(x_0)$. Hence we can easily find an $\tilde{E}_m \leq 0$ with $\tilde{E}_m - E_0 \sim 2^{-2m}S(x_0)$ and $\min_{k \in \mathbb{Z}} |\phi(\tilde{E}_m) - \pi(k + \frac{1}{2})| > \frac{1}{20}$. Lemma 1(B) from the section The Density for a One-Dimensional Potential in [FS3] shows that $V(x) - \tilde{E}_m, S_m, B_m, I_m$ satisfy hypotheses (Y0)–(Y11) for a suitable $\varphi_m(E)$. (For hypotheses (Y0)–(Y11), see [FS3].) Therefore, again applying Lemma 1 from the section Combining the Microlocalized Results in [FS3], we get the conclusion of Lemma 2. ■

LEMMA 3. For a suitable \tilde{E} with $\tilde{E} - V(x_0) \sim \lambda^{-2e}(x_0)S(x_0)$, and with $\min_{k' \in \mathbb{Z}} |\phi(\tilde{E}) - \pi(k' + \frac{1}{2})| \geq \frac{1}{20}$, the hypotheses (H0*)–(H6*) of the WKB Theorem on Low Eigenvalues are satisfied, with $V(x) - \tilde{E}$ in place of $V(x)$; with $S(x_0)$ in place of S ; with $B(x_0)$ in place of B ; and with $100K$ in place of K . The constants called c, c', c'', C_x in (H0*)–(H6*) all have the form $C_{\#}$. The number called λ in (H0*)–(H6*) is equal to $\lambda(x_0)$.

Proof. For $E - V(x_0) \sim \lambda^{-2e}(x_0)S(x_0)$ we have $d\phi(E)/dE \sim \lambda(x_0)S^{-1}(x_0)$. Hence we can pick an \tilde{E} with $\tilde{E} - V(x_0) \sim \lambda^{-2e}(x_0)S(x_0)$ and $\min_{k' \in \mathbb{Z}} |\phi(\tilde{E}) - \pi(k' + \frac{1}{2})| \geq \frac{1}{20}$. With $S = S(x_0)$, $B = B(x_0)$ and with $V(x) - \tilde{E}$ in place of $V(x)$, we check (H0*)–(H6*). Hypothesis (H0*) is trivial from $x_0 \in [x_{\text{left}}(0), x_{\text{rt}}(0)] \subset I$, $\text{dist}(x_{\text{left}}(0), \partial I) > cB(x_{\text{left}}(0))$, $\text{dist}(x_{\text{rt}}(0), \partial I) > cB(x_{\text{rt}}(0))$. Hypothesis (H1*) is trivial from (Z1); (H2*), (H3*) are trivial from (Z3).

To prove (H4*), we argue as follows. Since $\tilde{E} - V(x_0) \sim \lambda^{-2e}(x_0)S(x_0)$, we have $V(x) = \tilde{E}$ at $x = x_{\text{left}}(\tilde{E})$ and at $x = x_{\text{rt}}(\tilde{E})$, with $x_{\text{left}}(\tilde{E}) < x_0 < x_{\text{rt}}(\tilde{E})$, $x_{\text{left}}(\tilde{E})$ and $x_{\text{rt}}(\tilde{E})$ in $\{|x - x_0| < C_{\#}\lambda^{-e}(x_0)B(x_0)\}$. Now $V(x)$ is decreasing in $[x_{\text{left}}(0), x_{\text{left}}(\tilde{E})]$ and positive in $I_{\text{BVP}} \cap (-\infty, x_{\text{left}}(0)]$, so $V(x) > \tilde{E}$ for $x \in I_{\text{BVP}}$ with $x < x_{\text{left}}(\tilde{E})$. Similarly, $V(x) > \tilde{E}$ for $x \in I_{\text{BVP}}$ with $x > x_{\text{rt}}(\tilde{E})$. Hence, $V(x) - \tilde{E} > 0$ for $x \in I_{\text{BVP}} \setminus \{|x - x_0| < C_{\#}\lambda^{-e}(x_0)B(x_0)\}$, which is stronger than (H4*) since we take $E_{\infty} = 0$.

To prove (H5*), we pick E_+ with $E_+ - \tilde{E} \sim \tilde{E} - V(x_0)$. The proof of (H4*) above shows that $V(x) > E_+$ outside $[x_{\text{left}}(E_+), x_{\text{rt}}(E_+)]$. For $|x - x_0| > \frac{1}{2}\lambda^K(x_0)B(x_0)$ we have $x \notin [x_{\text{left}}(E_+), x_{\text{rt}}(E_+)]$, so $V(x) - \tilde{E} \geq E_+ - \tilde{E} \geq \lambda^{-2e}(x_0)S(x_0) \geq 1000/|x - x_0|^2$ as needed.

To see the last inequality, we write $1000/|x - x_0|^2 \leq 4000/\lambda^{2K}(x_0) B^2(x_0) = 4000S(x_0)/\lambda^{2K+2}(x_0) \leq \lambda^{-2\epsilon}(x_0) S(x_0)$. The proof of (H5*) is complete.

Finally, (H6*) is immediate from $\lambda(x_0) \geq c_{\#} A$ and the WKB hypothesis (Z9). The proof of Lemma 3 is complete. ■

We want to use the above lemmas, together with the WKB Eigenvalue Theorem and the WKB Theorem on Low Eigenvalues, to give a precise description of the negative eigenvalues of $-d^2/dx^2 + V(x)$. To carry this out, and also to be able to apply the lemma on Riemann sums, we need to understand how the phases $\phi(E)$, $\psi(E)$ vary with E . For $V(x_0) < E \leq 0$, define

$$S_{\min}(E) = \inf\{S(x) \mid x \in I, V(x) < E\} \quad (1)$$

and

$$\Gamma(E) = S_{\min}(E) \cdot \int_I (E - V(x))_+^{-1/2} dx. \quad (2)$$

For $E > 0$, define $S_{\min}(E) = S_{\min}(0)$ and $\Gamma(E) = \Gamma(0)$. Note that

$$\begin{aligned} S_{\min}(E) &\sim S(x_0) \text{ and } \Gamma(E) \sim \lambda(x_0) \quad \text{when } V(x_0) < E < V(x_0) \\ &+ c_{\#} S(x_0). \end{aligned} \quad (3)$$

Also, note that

$$\Gamma(E) \geq c_{\#} A \quad \text{for all } E \in (V(x_0), \infty). \quad (4)$$

To see this, we need only consider $E \in (V(x_0) + c_{\#} S(x_0), 0]$, in view of (3) and the definition of $\Gamma(E)$ for $E > 0$. For such E , we have $S_{\min}(E) \sim S(x_*)$ for some x_* with $[x_* - c_{\#} B(x_*), x_* + c_{\#} B(x_*)] \subset \{x \in I \mid V(x) < E\}$. Hence (2) gives $\Gamma(E) \geq [c_{\#} S(x_*)] \int_{|x - x_*| < c_{\#} B(x_*)} [C_{\#} S(x_*)]^{-1/2} dx = c'_{\#} S^{1/2}(x_*) B(x_*) = c'_{\#} \lambda(x_*) \geq c''_{\#} A$, proving (4). We have used (4) several times already in [FS3]. For $E \in (V(x_0), V(x_0) + c_{\#} S_{\min}(0))$ we have

$$\left| \left(\frac{d}{dE} \right)^{\beta} \phi(E) \right| \leq C_{\#}^{\beta} \Gamma(E) S_{\min}^{-\beta}(E) \quad (\beta \geq 1) \quad (5)$$

$$\left| \left(\frac{d}{dE} \right)^{\beta} \psi(E) \right| \leq C_{\#}^{\beta} A^{-2} \Gamma(E) S_{\min}^{-\beta}(E) \quad (\beta \geq 1) \quad (6)$$

$$\left[\left(\frac{d}{dE} \right) \phi(E) \right] \geq c_{\#} \Gamma(E) S_{\min}^{-1}(E). \quad (7)$$

In fact, for $E < V(x_0) + c_{\#} S(x_0)$, estimates (5) and (6) are contained in Lemma 6 in the section of [FS2] on eigenvalues near the minimum of the

potential, by virtue of (3) and Lemma 3 above. We leave the proof of (7) for $E < V(x_0) + c_{\#} S(x_0)$ to the reader. For $E_0 \in (V(x_0) + c_{\#} S(x_0), 0]$, we apply Lemma 1 above, and Lemma 1 in the section on the WKB theorems, to get

$$\left| \left(\frac{d}{dE} \right)^{\beta} \phi(E) \right| \leq C_{\#}^{\beta} \Gamma(E_0) S_{\min}^{-\beta}(E_0) \quad (\beta \geq 1) \quad (8)$$

$$\left| \left(\frac{d}{dE} \right)^{\beta} \psi(E) \right| \leq C_{\#}^{\beta} \Gamma(E_0) S_{\min}^{-\beta}(E_0) \Lambda^{-2} \quad (\beta \geq 1) \quad (9)$$

$$\frac{d\phi}{dE} \geq c_{\#} \Gamma(E_0) S_{\min}^{-1}(E_0). \quad (10)$$

for $|E - E_0| \leq c_{\#} S_{\min}(E_0)$.

Taking $E_0 = 0$ in (8), (9), (10) proves (5), (6), (7) for $E \in [0, c_{\#} S_{\min}(0)]$. Taking $E_0 = E$ in (8), (9), (10) gives (5), (6), (7) for $E \in [V(x_0) + c_{\#} S(x_0), 0]$, which completes the proof of (5), (6), (7).

Note that for $|E_1 - E_0| \leq c_{\#} S_{\min}(E_0)$, E_1 and $E_0 \in (V(x_0), c_{\#} S_{\min}(0)]$, we have

$$S_{\min}(E_1) \sim S_{\min}(E_0) \quad \text{and} \quad \Gamma(E_1) \sim \Gamma(E_0). \quad (11)$$

This is easily verified by looking separately at the cases $E_0 \in (V(x_0), V(x_0) + c_{\#} S(x_0)]$ and $E_0 \in (V(x_0) + c_{\#} S(x_0), c_{\#} S_{\min}(0)]$. From (5), (6), (7) we see that

$$E \mapsto \Phi(E) \equiv \frac{1}{\pi} \phi(E) + \frac{1}{48\pi} \psi(E) - \frac{1}{2} \quad (12)$$

is strictly increasing on $(V(x_0), c_{\#} S_{\min}(0)]$.

From (5) to (11) follow these quantitative results:

$$\begin{aligned} &\text{Suppose } E_1 \geq E_0 + c_{\#} S_{\min}(E_0), \text{ with } E_0, E_1 \in (V(x_0), c_{\#} S_{\min}(0)]. \\ &\text{Then } \Phi(E_1) \geq \Phi(E_0) + c'_{\#} \Gamma(E_0). \end{aligned} \quad (13)$$

$$\begin{aligned} &\text{Suppose } E_1 \leq E_0 - c_{\#} S_{\min}(E_0), \text{ with } E_0, E_1 \in (V(x_0), c_{\#} S_{\min}(0)]. \\ &\text{Then } \Phi(E_1) \leq \Phi(E_0) - c'_{\#} \Gamma(E_0). \end{aligned} \quad (14)$$

From (13) and (14) follows at once

$$\begin{aligned} &\text{Suppose } E_0, E_1 \in (V(x_0), c_{\#} S_{\min}(0)] \text{ and } |\Phi(E_0) - \Phi(E_1)| < \\ &c'_{\#} \Gamma(E_0). \\ &\text{Then } |E_1 - E_0| \leq c_{\#} S_{\min}(E_0). \end{aligned} \quad (15)$$

For $|E - E_0| \leq c_{\#} S_{\min}(E_0)$ we have $\Phi'(E) \sim S_{\min}^{-1}(E_0) \Gamma(E_0)$ by (11) and (5), (6), (7). This and (15) yield the following.

Suppose $E_0, E_1 \in (V(x_0), c_{\#} S_{\min}(0)]$ and $|\Phi(E_0) - \Phi(E_1)| \leq c'_{\#} \Gamma(E_0)$.

Then $|E_0 - E_1| \sim S_{\min}(E_0) \Gamma^{-1}(E_0) \cdot |\Phi(E_0) - \Phi(E_1)|$. (16)

Now we can start to control the eigenvalues of $-d^2/dx^2 + V(x)$. Let k_{hi} be the greatest integer $\leq \Phi(0) + \bar{C}_{\#} \Lambda^{4\epsilon-2}$ for a suitable, large $\bar{C}_{\#}$.

LEMMA 4. For $0 \leq k \leq k_{hi} - 1$, there is an eigenvalue $E \leq 0$ with $|\Phi(E) - k| \leq C_{\#} \Lambda^{4\epsilon-2}$.

Proof. As E varies from $V(x_0)$ to 0, $\Phi(E)$ varies from $-\frac{1}{2} + O(\Lambda^{-1}) < k$ to $\Phi(0) \geq k_{hi} - \bar{C}_{\#} \Lambda^{4\epsilon-2} > k_{hi} - 1 \geq k$. Hence there is some $E_0 \in (V(x_0), 0)$ for which $\Phi(E_0) = k$.

We distinguish three cases:

- (A) $E_0 - V(x_0) \geq c_{\#} S(x_0)$
- (B) $c_{\#} \lambda^{-2\epsilon} S(x_0) \leq E_0 - V(x_0) \leq c_{\#} S(x_0)$
- (C) $E_0 - V(x_0) \leq c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)$.

First suppose we are in Case (A). Lemma 1 shows that the WKB Eigenvalue Theorem applies. Note that our present phase function $\Phi(E)$ is slightly different from the phase function called $\Phi(E)$ in the WKB Eigenvalue Theorem, which we now call $\Phi_{\text{WKB}}(E)$. In fact, $\Phi_{\text{WKB}}(E)$ is defined for $E \in [E_0 - c_{\#} S_{\min}(E_0), E_0 + c_{\#} S_{\min}(E_0)] \cap (-\infty, 0]$ and satisfies there $|\Phi_{\text{WKB}}(E) - \pi\Phi(E)| \leq C_{\#} \Lambda^{-2}$. This is a part of the conclusion of the WKB Eigenvalue Theorem. Let $E_+ = \min(0, E_0 + c_{\#} S_{\min}(E_0))$, $E_- = E_0 - c_{\#} S_{\min}(E_0)$. These are the endpoints of the interval on which Φ_{WKB} is defined.

We shall check that k lies in the image of $(1/\pi) \Phi_{\text{WKB}}$, by examining $\Phi_{\text{WKB}}(E_+)$ and $\Phi_{\text{WKB}}(E_-)$. Let us start with $\Phi_{\text{WKB}}(E_+)$. Either $E_+ = 0$ or $E_+ = E_0 + c_{\#} S_{\min}(E_0)$. If $E_+ = 0$, then

$$\frac{1}{\pi} \Phi_{\text{WKB}}(E_+) \geq \Phi(0) - C_{\#} \Lambda^{-2} \geq [k_{hi} - \bar{C}_{\#} \Lambda^{4\epsilon-2}] - C_{\#} \Lambda^{-2} > k_{hi} - 1 \geq k, \quad (16\text{bis})$$

so $(1/\pi) \Phi_{\text{WKB}}(E_+) > k$. If $E_+ = E_0 + c_{\#} S_{\min}(E_0)$, then (5), (6), (7) give $(1/\pi) \Phi_{\text{WKB}}(E_+) \geq \Phi(E_+) - C_{\#} \Lambda^{-2} \geq \Phi(E_0) + c'_{\#} \Gamma(E_0) - C_{\#} \Lambda^{-2} > \Phi(E_0) = k$, so again $(1/\pi) \Phi_{\text{WKB}}(E_+) > k$. Hence in all cases $(1/\pi) \Phi_{\text{WKB}}(E_+) > k$. On the other hand, $(1/\pi) \Phi_{\text{WKB}}(E_-) \leq \Phi(E_-) + C_{\#} \Lambda^{-2} \leq \Phi(E_0) - c'_{\#} \Gamma(E_0) + C_{\#} \Lambda^{-2} < \Phi(E_0) = k$, so $(1/\pi) \Phi_{\text{WKB}}(E_-) < k$. Therefore, $\pi k \in \Phi_{\text{WKB}}([E_0 - c_{\#} S_{\min}(E_0), E_0 + c_{\#} S_{\min}(E_0)] \cap (-\infty, 0])$, as claimed. The WKB Eigenvalue Theorem then asserts that

$$|\Phi_{\text{WKB}}(\tilde{E}) - \pi k| \leq C_{\#} \Lambda^{-N''},$$

where \tilde{E} is either an eigenvalue or 0. In particular

$$|\Phi(\tilde{E}) - k| \leq C_{\#} \Lambda^{-2} \quad (17)$$

Since

$$\Phi(0) \geq k_{hi} - \bar{C}_{\#} \Lambda^{4e-2} > (k_{hi} - 1) + \frac{1}{2} \geq k + \frac{1}{2}, \quad (18)$$

(17) cannot hold with $\tilde{E} = 0$. Hence \tilde{E} is an eigenvalue. Also, (17) and (18) yield $\Phi(\tilde{E}) < \Phi(0)$, so $\tilde{E} < 0$. We have proven (17) for a negative eigenvalue \tilde{E} . Thus the conclusion of Lemma 4 holds in Case (A).

Next suppose we are in Case (B). Define an integer m by

$$2^{-2(m+1)} c_{\#} S(x_0) < E_0 - V(x) \leq 2^{-2m} c_{\#} S(x_0). \quad (19)$$

Since we are in Case (B), we have

$$c_{\#} \lambda^{-2e}(x_0) \leq 2^{-2m} \leq 1. \quad (20)$$

Lemma 2 above lets us apply the WKB Eigenvalue Theorem to the potential $V(x) - \tilde{E}_m$, with weight functions $2^{-2m} S(x_0)$, $2^{-m} B(x_0)$ in place of $S(x)$, $B(x)$; with $\{|x - x_0| < C_{\#} \cdot 2^{-m} B(x_0)\}$ in place of I ; with $E_0 - \tilde{E}_m$ in place of E_0 ; with $E_{\infty} = 0$; and with $300K$ in place of K . Here, $\tilde{E}_m - E_0 \sim 2^{-2m} c_{\#} S(x_0)$. The WKB Eigenvalue Theorem tells us the following: There is a smooth phase function $\Phi_{\text{WKB}}(E)$ defined on $[E_0 - c'_{\#} 2^{-2m} S(x_0), E_0 + c'_{\#} 2^{-2m} S(x_0)]$ and satisfying there

$$|\Phi_{\text{WKB}}(E) - \pi\Phi(E)| \leq C_{\#} (2^{-2m} \lambda(x_0))^{-2}. \quad (21)$$

If πk belongs to $\Phi_{\text{WKB}}([E_0 - c''_{\#} 2^{-2m} S(x_0), E_0 + c''_{\#} 2^{-2m} S(x_0)] \cap (-\infty, \tilde{E}_m])$ then we can find \tilde{E} either an eigenvalue of $-d^2/dx^2 + V(x)$ or \tilde{E}_m , satisfying

$$|\Phi_{\text{WKB}}(\tilde{E}) - \pi k| \leq C_{\#} (2^{-2m} \lambda(x_0))^{-N''}. \quad (22)$$

Since $\tilde{E}_m - E_0 \sim 2^{-2m} c_{\#} S(x_0)$, we have

$$\begin{aligned} & [E_0 - c''_{\#} S(x_0) 2^{-2m}, E_0 + c''_{\#} S(x_0) 2^{-2m}] \cap (-\infty, \tilde{E}_m] \\ &= [E_-, E_+] \subset (V(x_0), c_{\#} S_{\min}(0)) \end{aligned} \quad (23)$$

with

$$(E_0 - E_-), (E_+ - E_0) \sim 2^{-2m} S(x_0). \quad (24)$$

From (19), (20) we get $\Gamma(E_0) \sim \lambda(x_0)$, $S_{\min}(E_0) \sim S(x_0)$, so $\Phi'(E) \sim \Gamma(E_0) S_{\min}^{-1}(E_0) \sim \lambda(x_0) S^{-1}(x_0)$ for $|E - E_0| < c_{\#} S(x_0)$, hence for $E \in [E_-, E_+]$. Therefore, (24) implies

$$\Phi(E_0) - \Phi(E_-) \sim 2^{-2m} \lambda(x_0), \quad \Phi(E_+) - \Phi(E_0) \sim 2^{-2m} \lambda(x_0). \quad (25)$$

Note that $2^{-2m} \lambda(x_0) \geq c_{\#} \lambda^{1-2\epsilon}(x_0) \gg 1$ by (20), and recall that $\Phi(E_0) = k$. Then (21) and (25) imply $\Phi_{\text{WKB}}(E_+) > \pi k > \Phi_{\text{WKB}}(E_-)$, so that $\pi k \in \Phi_{\text{WKB}}([E_-, E_+])$. Thus we can find $\tilde{E} = \tilde{E}_m$ or an eigenvalue, satisfying (22). From (21) and (22), we get

$$|\Phi(\tilde{E}) - k| \leq C_{\#} (2^{-2m} \lambda(x_0))^{-2}. \quad (26)$$

In particular, since $\Phi(E_0) = k$, we see from (25), (26) that $\Phi(\tilde{E}) < \Phi(E_+)$, hence $\tilde{E} < E_+ \leq \tilde{E}_m$. This shows that $\tilde{E} \neq \tilde{E}_m$, so \tilde{E} is an eigenvalue of $-d^2/dx^2 + V(x)$.

Also, since $\tilde{E} \leq \tilde{E}_m < 0$, we get $\tilde{E} < 0$. Hence (26) holds for a negative eigenvalue \tilde{E} . From (26) and (20), we get $|\Phi(\tilde{E}) - k| \leq C_{\#} (\lambda^{1-2\epsilon}(x_0))^{-2} \leq C'_{\#} \Lambda^{4\epsilon-2}$. Thus, the conclusion of Lemma 4 holds in Case (B).

Finally, suppose we are in Case (C). Lemma 3 above lets us apply the WKB Theorem on Low Eigenvalues to the potential $V(x) - \tilde{E}$, with $S(x_0)$, $B(x_0)$ in place of S, B ; with $E_{\infty} = 0$; and with $100K$ in place of K . Here, $\tilde{E} - E_0 \sim \lambda^{-2\epsilon}(x_0) S(x_0)$, and $\tilde{E} < 0$. The WKB Theorem on Low Eigenvalues provides the following information:

Suppose $(1/\pi) \phi(\min\{\tilde{E}, V(x_0) + c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)\}) - \frac{1}{2} \geq k + 1$. Then there is an eigenvalue $E_k \leq \tilde{E}$ with

$$|\Phi(E_k) - k| \leq C_{\#} (\lambda(x_0))^{4\epsilon-2}. \quad (27)$$

To check the condition on k , we note that $V(x_0) + c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0) \geq E_0 + c'_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)$, provided we take the constant $c_{\#}$ in the definition of Case (C) small enough. Since also $\tilde{E} \geq E_0 + c''_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)$, we have

$$E_+ = \min\{\tilde{E}, V(x_0) + c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)\} \geq E_0 + c'''_{\#} \lambda^{-2\epsilon}(x_0) S(x_0). \quad (28)$$

We have $V(x_0) < E_0 \leq E_+ \leq V(x_0) + c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)$, so $\phi'(E) \sim \lambda(x_0) S^{-1}(x_0)$ for E between E_0 and E_+ . Hence, (28) implies

$$\begin{aligned} \left[\frac{1}{\pi} \phi(E_+) - \frac{1}{2} \right] &\geq \left[\frac{1}{\pi} \phi(E_0) - \frac{1}{2} \right] + c_{\#} \lambda^{1-2\epsilon}(x_0) \\ &= \Phi(E_0) - \frac{1}{48\pi} \psi(E_0) + c_{\#} \lambda^{1-2\epsilon}(x_0) \geq \Phi(E_0) + c'_{\#} \lambda^{1-2\epsilon}(x_0) \end{aligned}$$

(since $|\psi(E_0)| \leq C_{\#} \Lambda^{-1} \leq 1 \leq \lambda^{1-2\epsilon}(x_0) = k + c'_{\#} \lambda^{1-2\epsilon}(x_0)$). This verifies the condition on k , which implies (27) for an eigenvalue $E_k \leq \tilde{E} < 0$.

From (27) we get in particular $|\Phi(E_k) - k| \leq C_{\#} \Lambda^{4\epsilon-2}$ for an eigenvalue $E_k < 0$. Thus the conclusion of Lemma 4 holds also in Case (C). The proof of Lemma 4 is complete. ■

COROLLARY. *If $\Phi(0) \geq k_{hi} + \bar{C}_{\#} \Lambda^{4\epsilon-2}$, then there is an eigenvalue $E \leq 0$ with $|\Phi(E) - k_{hi}| \leq \bar{C}_{\#} \Lambda^{4\epsilon-2}$.*

Proof. As E varies from $V(x_0)$ to 0, $\Phi(E)$ varies from $-\frac{1}{2} + O(\Lambda^{-1}) < k_{hi}$ to $\Phi(0) > k_{hi}$. Hence there is some $E_0 \in (V(x_0), 0)$ for which $\Phi(E_0) = k_{hi}$.

Now we can repeat the proof of Lemma 4, using k_{hi} in place of k , and making the following modifications. Instead of (16bis), we note that $(1/\pi) \Phi_{\text{WKB}}(E_+) \geq \Phi(0) - C_{\#} \Lambda^{-2} \geq [k_{hi} + \bar{C}_{\#} \Lambda^{4\epsilon-2}] - C_{\#} \Lambda^{-2} > k_{hi}$. Instead of (18), we note that $\Phi(0) \geq k_{hi} + \bar{C}_{\#} \Lambda^{4\epsilon-2}$.

With these minor changes the proof of Lemma 4 goes through. ■

LEMMA 5. *If $E_0 \leq 0$ is an eigenvalue of $-d^2/dx^2 + V(x)$, then $|\Phi(E_0) - k| \leq \bar{C}_{\#} \Lambda^{4\epsilon-2}$ for an integer k ($0 \leq k \leq k_{hi}$).*

Proof. We distinguish three cases:

- (A) $E_0 - V(x_0) \geq c_{\#} S(x_0)$
- (B) $c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0) \leq E_0 - V(x_0) \leq c_{\#} S(x_0)$
- (C) $E_0 - V(x_0) \leq c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)$.

First suppose we are in Case (A). Lemma 1 and the WKB Eigenvalue Theorem show that $|\Phi(E) - (\text{integer})| \leq C_{\#} \Lambda^{-2}$ for any eigenvalue $E \in [E_0 - c'_{\#} S_{\min}(E_0), E_0 + c'_{\#} S_{\min}(E_0)] \cap (-\infty, 0]$. In particular, $|\Phi(E_0) - (\text{integer})| \leq C_{\#} \Lambda^{-2}$.

Next, suppose we are in Case (B). Define an integer m by $2^{-2(m+1)} c_{\#} S(x_0) < E_0 - V(x) \leq 2^{-2m} c_{\#} S(x_0)$. Since we are in Case (B), we have $c_{\#} \lambda^{-2\epsilon}(x_0) \leq 2^{-2m} \leq 1$. Lemma 2 and the WKB Eigenvalue Theorem show that $|\Phi(E) - (\text{integer})| \leq C_{\#} (2^{-2m} \lambda(x_0))^{-2}$ for any eigenvalue E in the interval $[E_0 - c'_{\#} (2^{-2m} S(x_0)), E_0 + c'_{\#} (2^{-2m} S(x_0))] \cap (-\infty, \tilde{E}_m]$. (Here $\tilde{E}_m > E_0$ is as in Lemma 2.) In particular, for E_0 itself we have $|\Phi(E_0) - (\text{integer})| \leq C_{\#} (2^{-2m} \lambda(x_0))^{-2} \leq C_{\#} (\lambda^{1-2\epsilon}(x_0))^{-2} \leq C_{\#} \Lambda^{4\epsilon-2}$.

Next, suppose we are in Case (C). We apply Lemma 3 and the WKB Theorem for Low Eigenvalues. This shows that $|\Phi(E) - k| \leq C_{\#} (\lambda(x_0))^{4\epsilon-2}$ for any eigenvalue E satisfying

$$E \leq \min\{\tilde{E}, V(x_0) + c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)\}. \quad (29)$$

(Here $\tilde{E} > E_0$ is as in Lemma 3.) In particular, $E = E_0$ satisfies (29), provided we pick $c_{\#}$ small enough in the statement of Case (C). Hence,

$|\Phi(E_0) - (\text{integer})| \leq C_{\#} (\lambda(x_0))^{4\epsilon-2} \leq C_{\#} A^{4\epsilon-2}$. We have proven in all cases that

$$|\Phi(E_0) - k| \leq C_{\#} A^{4\epsilon-2} \quad \text{for an integer } k. \quad (30)$$

As E varies from $V(x_0)$ to 0, $\Phi(E_0)$ varies from $-\frac{1}{2} + O(\lambda^{-1}(x_0))$ to $\Phi(0)$. Hence (30) implies

$$-\frac{1}{2} - C_{\#} \lambda^{-1}(x_0) - C_{\#} A^{4\epsilon-2} \leq k \leq \Phi(0) + C_{\#} A^{4\epsilon-2}. \quad (31)$$

Since k is an integer, (31) implies

$$0 \leq k \leq \Phi(0) + \bar{C}_{\#} A^{4\epsilon-2}, \quad (32)$$

provided we take $\bar{C}_{\#}$ larger than the $C_{\#}$ in (31). Then from (32) and the definition of k_{hi} , we see that $0 \leq k \leq k_{hi}$. Thus, $|\Phi(E_0) - k| \leq \bar{C}_{\#} A^{4\epsilon-2}$ and $0 \leq k \leq k_{hi}$, provided we also take $\bar{C}_{\#}$ larger than the $C_{\#}$ in (30). The proof of Lemma 5 is complete. ■

LEMMA 6. Suppose $\bar{E}_0, \bar{E}_1 \leq 0$ are eigenvalues of $-d^2/dx^2 + V(x)$. If $|\Phi(\bar{E}_0) - \Phi(\bar{E}_1)| \leq C_{\#} A^{4\epsilon-2}$ then $\bar{E}_0 = \bar{E}_1$.

Proof. Suppose not. We may suppose $\bar{E}_1 < \bar{E}_0 \leq 0$. Then (16) implies

$$\bar{E}_0 - C_{\#} S_{\min}(\bar{E}_0) \Gamma^{-1}(\bar{E}_0) A^{4\epsilon-2} \leq \bar{E}_1 < \bar{E}_0 \leq 0. \quad (33)$$

We distinguish three cases:

- (A) $\bar{E}_0 - V(x_0) \geq c_{\#} S(x_0)$
- (B) $c_{\#} \lambda^{-2\epsilon} S(x_0) \leq \bar{E}_0 - V(x_0) \leq c_{\#} S(x_0)$
- (C) $\bar{E}_0 - V(x) \leq c_{\#} \lambda^{-2\epsilon}(x_0) S(x_0)$.

Suppose first we are in Case (A). Take $E_0 = \bar{E}_0$, and apply Lemma 1 and the WKB Eigenvalue Theorem. Thus, the eigenvalues in $\mathcal{E} = [\bar{E}_0 - c'_{\#} S_{\min}(\bar{E}_0), \bar{E}_0 + c'_{\#} S_{\min}(\bar{E}_0)] \cap (-\infty, 0]$ are among the E_k described in the WKB Eigenvalue Theorem, and distinct E_k have $|\Phi(E_k) - \Phi(E_{k'})| \geq 1 - C_{\#} A^{-2}$. Since (33) shows that $\bar{E}_0, \bar{E}_1 \in \mathcal{E}$, it follows that $|\Phi(\bar{E}_0) - \Phi(\bar{E}_1)| \geq 1 - C_{\#} A^{-2}$, contradicting the hypothesis of the lemma.

Next suppose we are in Case (B). Define an integer m by $2^{-(m+1)} c_{\#} S(x_0) < \bar{E}_0 - V(x_0) \leq 2^{-2m} c_{\#} S(x_0)$. Since we are in Case (B), we have

$$c_{\#} \lambda^{-2\epsilon}(x_0) < 2^{-2m} \leq 1. \quad (34)$$

Take $E_0 = \bar{E}_0$, and apply Lemma 2 and the WKB Eigenvalue Theorem. Thus, the eigenvalues in $\mathcal{E} \equiv [\bar{E}_0 - c'_\# 2^{-2m} S(x_0), \bar{E}_0 + c'_\# 2^{-2m} S(x_0)] \cap (-\infty, \tilde{E}_m]$ are among the E_k described in the WKB theorem, so that distinct E_k have $|\Phi(E_k) - \Phi(E_{k'})| \geq 1 - C_\# \cdot (2^{-2m} \lambda(x_0))^{-2}$. (Here $\tilde{E}_m > \bar{E}_0$ is as in Lemma 2.) In particular, (33) and (3) show that $\bar{E}_0 - C_\# S(x_0) \lambda^{-1}(x_0) A^{4e-2} \leq \bar{E}_1 < \bar{E}_0 < \tilde{E}_m \leq 0$, which implies $\bar{E}_0, \bar{E}_1 \in \mathcal{E}$ by virtue of (34). Therefore, $|\Phi(\bar{E}_0) - \Phi(\bar{E}_1)| \geq 1 - C_\# (2^{-2m} \lambda(x_0))^{-2} \geq 1 - C'_\# \lambda^{4e-2}(x_0) > \frac{1}{2}$, contradicting the hypothesis of the lemma.

Finally, suppose we are in Case (C). We apply Lemma 3 and the WKB Theorem on Low Eigenvalues. Thus, all the eigenvalues in $\mathcal{E} = [V(x_0), V(x_0) + c_\# \lambda^{-2e}(x_0) S(x_0)] \cap (-\infty, \tilde{E}]$ are among the E_k described in the WKB Theorem on Low Eigenvalues, and distinct E_k satisfy $|\Phi(E_k) - \Phi(E_{k'})| \geq 1 - C_\# \lambda^{4e-2}(x_0)$. Here $\tilde{E} \leq 0$ is as in Lemma 3, so that $\tilde{E} - V(x_0) \sim \lambda^{-2e}(x_0) S(x_0)$. Since we are in Case (C), we have $\bar{E}_0 < \tilde{E}$ and $\bar{E}_0 < V(x_0) + c_\# \lambda^{-2e}(x_0) S(x_0)$, provided we take the $c_\#$ in the statement of Case (C) small enough. Thus $\bar{E}_0 \in \mathcal{E}$. Since $V(x_0) < \bar{E}_1 < \bar{E}_0$ we have also $\bar{E}_1 \in \mathcal{E}$. Hence $|\Phi(\bar{E}_0) - \Phi(\bar{E}_1)| \geq 1 - C_\# \lambda^{4e-2}(x_0) > \frac{1}{2}$, contradicting the hypothesis of the lemma.

In all three cases (A), (B), (C), we have reached a contradiction, starting with the assumption $\bar{E}_1 < \bar{E}_0$. The proof of Lemma 6 is complete. ■

LEMMA 7. *The non-positive eigenvalues of $-d^2/dx^2 + V(x)$ are $E_0, E_1, \dots, E_{k_{\max}}$, with*

$$|\Phi(E_k) - k| \leq \bar{C}_\# A^{4e-2}, \quad (35)$$

$$k_{\max} = \text{greatest integer} \leq \Phi(0) + w_{hi}, \quad (36)$$

and

$$|w_{hi}| \leq \bar{C}_\# A^{4e-2}. \quad (37)$$

Proof. This follows from Lemma 4 and its corollary, and from Lemmas 5 and 6. We take $k_{\max} = k_{hi}$ or $k_{hi} - 1$, depending on whether there is a non-positive eigenvalue E satisfying $|\Phi(E) - k_{hi}| \leq \bar{C}_\# A^{4e-2}$. For $0 \leq k \leq k_{\max}$, we can find exactly one non-positive eigenvalue E_k satisfying (35). (That follows from Lemmas 4 and 6.) Moreover, every non-positive eigenvalue is one of the E_k , by Lemma 5. If $k_{\max} = k_{hi}$, then we take $w_{hi} = \bar{C}_\# A^{4e-2}$, so that (36) and (37) hold by definition of k_{hi} and w_{hi} . If instead $k_{\max} = k_{hi} - 1$, then

$$\Phi(0) < k_{hi} + \bar{C}_\# A^{4e-2} \quad (38)$$

by the corollary to Lemma 4. Hence we may set $w_{hi} = -\bar{C}_\# A^{4e-2}$, so that (37) holds by definition and (36) amounts to saying that

$$k_{hi} - 1 \leq \Phi(0) - \bar{C}_\# A^{4e-2} < k_{hi}. \quad (39)$$

Since $k_{hi} \leq \Phi(0) + \bar{C}_\# A^{4\epsilon-2}$ by definition of k_{hi} , the first inequality in (39) is obvious. The second inequality in (39) is the known fact (38). The proof of Lemma 7 is complete. ■

COROLLARY. *The eigenvalues of $-d^2/dx^2 + V(x)$ that belong to $[E_*, 0]$ are precisely the E_k for $\Phi(E_*) \leq k \leq \Phi(0) + w_{hi}$.*

Proof. We must check that $E_k \in [E_*, 0]$ if and only if $k \geq \Phi(E_*)$. Since $\Phi(\cdot)$ is strictly monotone, $E_k \geq E_*$ if and only if $\Phi(E_k) \geq \Phi(E_*)$. We know that $\Phi(E_k)$ is near k , and that $\Phi(E_*) = (1/\pi) \phi(E_*) + O(A^{-1}) - \frac{1}{2}$ is near $(\text{integer} - \frac{1}{2})$, hence not too near any integer. (Here we use our assumption that $(1/\pi) \phi(E_*)$ is near an integer.) It follows that $\Phi(E_k) \geq \Phi(E_*)$ if and only if $k \geq \Phi(E_*)$. Thus, $E_k \geq E_*$ if and only if $k \geq \Phi(E_*)$. ■

We prepare to use the above corollary and the lemma on Riemann sums to compute the sum of the eigenvalues $E_k \in [E_*, 0]$. Define $(t_{\min}, t_{\max}] = \text{Image of } \Phi \text{ on } (V(x_0), c_\# S_{\min}(0)]$. For $t \in (t_{\min}, t_{\max}]$, define $E(t)$ to be the solution E of the equation $\Phi(E) = t$; and define $\sigma(t) = S_{\min}(E(t))$, $\tau(t) = \Gamma(E(t))$. Estimates (5), (6), (7) show that

$$\left| \left(\frac{d}{dt} \right)^m E(t) \right| \leq C_\#^m \sigma(t) \tau^{-m}(t) \quad \text{for } t \in (t_{\min}, t_{\max}], \quad (40)$$

when $m \geq 1$. We check that (40) holds also for $m=0$, which amounts to saying that

$$|E| \leq C_\# S_{\min}(E) \quad \text{for } E \in (V(x_0), +c_\# S_{\min}(0)]. \quad (41)$$

When $E \in [0, +c_\# S_{\min}(0)]$, then (41) holds since $S_{\min}(E) = S_{\min}(0)$. When $E \in (V(x_0), 0]$, then for $x \in I$, $V(x) < E$ we have $-C_\# S(x) \leq V(x) < E \leq 0$, so $|E| \leq C_\# S(x)$. Hence, $|E| \leq C_\# \inf\{S(x) \mid x \in I, V(x) < E\} = c_\# S_{\min}(E)$, completing the proof of (41). Thus, (40) holds for all $m \geq 0$. Note also that

$$\sigma(t_1) \sim \sigma(t_0) \quad \text{and} \quad \tau(t_1) \sim \tau(t_0) \quad \text{for } t_0, t_1 \in (t_{\min}, t_{\max}], |t_0 - t_1| < c_\# \tau(t_0). \quad (42)$$

This follows at once from (11) and (15). From (4) we get

$$\tau(t) \geq c_\# A \quad \text{for } t \in (t_{\min}, t_{\max}]. \quad (43)$$

From (5), (6), (7) follows also

$$\frac{dE(t)}{dt} \geq c_\# \sigma(t) \tau^{-1}(t) \quad \text{for } t \in (t_{\min}, t_{\max}]. \quad (43\text{bis})$$

Because of (8), (9), (10) with $E_0=0$, we have $t_{\max} = \Phi(c_{\#} S_{\min}(0)) \geq \Phi(0) + c'_{\#} \Gamma(0) \geq \Phi(0) + c''_{\#} \Lambda$ (by (4)). Therefore, setting

$$a = \Phi(E_{*}), \quad b = \Phi(0) + w_{hi} \quad (44)$$

with w_{hi} as in Lemma 7, $b_0 = \Phi(0)$, and recalling that $|w_{hi}| \ll 1 \ll c''_{\#} \Lambda$, we obtain

$$t_{\min} < a \leq b < t_{\max}, \quad \text{and} \quad a \leq b_0 < t_{\max}. \quad (45)$$

(To see that $a \leq b$, b_0 we use the assumption $\phi(E_{*}) \leq \phi(0) - 1$, which we made at the start of this section.)

By definition, $S_{\min}(E)$ is constant for $E \geq 0$ and monotone decreasing for $E \leq 0$. Hence,

$$S_{\min}(E) \leq S_{\min}(E_{*}) \quad \text{for} \quad E \geq E_{*}, \quad (45\text{bis})$$

so that

$$\sigma(t) \leq S_{\min}(E_{*}) \quad \text{for} \quad t \in [a, b]. \quad (46)$$

Let $\{E_k\}$ be the non-positive eigenvalues of $-d^2/dx^2 + V(x)$, as in Lemma 7. Then (35) and (40) imply

$$\begin{aligned} |E_k - E(k)| &= |E(\Phi(E_k)) - E(k)| \leq |\Phi(E_k) - k| \\ &\cdot \sup \left\{ \left| \frac{dE(t)}{dt} \right| : |t - k| \leq |\Phi(E_k) - k| \right\} \\ &\leq \bar{C}_{\#} \Lambda^{4e-2} \cdot \sup \{ C_{\#} \sigma(t) \tau^{-1}(t) : |t - k| \leq \bar{C}_{\#} \Lambda^{4e-2} \}. \end{aligned}$$

Now (42), (43) imply that

$$\sigma(t) \tau^{-1}(t) \text{ has constant order of magnitude for } |t - k| \leq 1. \quad (48)$$

Putting this into (47) yields

$$|E_k - E(k)| \leq C_{\#} \Lambda^{4e-2} \sigma(k) \tau^{-1}(k). \quad (49)$$

Putting (48) into (43bis), we get also

$$\begin{aligned} \int_{[a, b_0] \cap [k-1, k+1]} \left(\frac{dE(t)}{dt} \right) dt &\geq c_{\#} \sigma(k) \tau^{-1}(k) \\ &\cdot \text{length}([a, b_0] \cap [k-1, k+1]) \end{aligned} \quad (50)$$

for $k \in \mathbb{Z} \cap [a, b]$. Our assumption $\phi(E_{*}) < \phi(0) - 1$ implies that $[a, b_0]$ has length $\geq c_{\#}$, so $\text{length}([a, b_0] \cap [k-1, k+1]) \geq c'_{\#}$ provided

$\text{distance}(k, [a, b_0]) \leq \bar{C}_\# A^{4\epsilon-2}$. For $k \in \mathbb{Z} \cap [a, b]$ we have $\text{dist}(k, [a, b_0]) \leq \bar{C}_\# A^{4\epsilon-2}$, since $|b - b_0| = |w_{hi}| \leq \bar{C}_\# A^{4\epsilon-2}$. Therefore, (50) implies

$$\int_{[a, b_0] \cap [k-1, k+1]} \left(\frac{dE(t)}{dt} \right) dt \geq c_\# \sigma(k) \tau^{-1}(k) \quad \text{for } k \in \mathbb{Z} \cap [a, b].$$

Combining this with (49), we find that

$$|E_k - E(k)| \leq C_\# A^{4\epsilon-2} \int_{[a, b_0] \cap [k-1, k+1]} \left(\frac{dE(t)}{dt} \right) dt \quad \text{for } k \in \mathbb{Z} \cap [a, b].$$

Summing on k , we get

$$\begin{aligned} \sum_{k \in \mathbb{Z} \cap [a, b]} |E_k - E(k)| &\leq C_\# A^{4\epsilon-2} \int_a^{b_0} \left(\frac{dE(t)}{dt} \right) dt \\ &= C_\# A^{4\epsilon-2} (E(b_0) - E(a)) = C_\# A^{4\epsilon-2} (0 - E_\star) \end{aligned}$$

(by definition of $a, b_0, E(\cdot)$). Thus

$$\sum_{k \in \mathbb{Z} \cap [a, b]} |E_k - E(k)| \leq C_\# |E_\star| A^{4\epsilon-2}. \quad (51)$$

We apply the Lemma on Riemann Sums to compute $\sum_{k \in \mathbb{Z} \cap [a, b]} E(k)$ modulo a small error. Estimates (40), (42), (43), (45) show that the hypotheses of the Lemma on Riemann Sums are satisfied, and therefore

$$\begin{aligned} \sum_{k \in \mathbb{Z} \cap [a, b]} E(k) &= \int_a^b E(t) dt - E(b) \chi_-(b) - E(a) \chi_+(a) \\ &\quad + \frac{1}{2} \frac{dE(t)}{dt} \Big|_{t=b} \tilde{\chi}(b) - \frac{1}{2} \frac{dE(t)}{dt} \Big|_{t=a} \tilde{\chi}(a) + \text{Error}_0, \end{aligned} \quad (52)$$

with

$$|\text{Error}_0| \leq C_\# \sigma(a) \tau^{-2}(a) + C_\# \sigma(b) \tau^{-2}(b) + C_\# \int_a^b \sigma(t) \tau^{-N}(t) dt. \quad (53)$$

Let us estimate $|\text{Error}_0|$. We have

$$\int_a^b \sigma(t) \tau^{-1}(t) dt \leq C_\# \int_a^b \frac{dE(t)}{dt} dt \quad (\text{by (43bis)}) = (E(b) - E(a)) C_\#.$$

Since $E(a) = E_\star$ and $E(b) \in (V(x_0), +c_\# S_{\min}(0)]$, it follows that

$$\int_a^b \sigma(t) \tau^{-1}(t) dt \leq C_\# S_{\min}(0) + C_\# |E_\star| \leq C_\# S_{\min}(E_\star),$$

by (41) and (45bis). Therefore,

$$\int_a^b \sigma(t) \tau^{-N}(t) dt \leq C_{\#} \Lambda^{1-N} S_{\min}(E_{\star}), \quad \text{by (43).} \quad (54)$$

Also,

$$\sigma(a) \tau^{-2}(a), \sigma(b) \tau^{-2}(b) \leq C_{\#} S_{\min}(E_{\star}) \cdot \Lambda^{-2}, \quad \text{by (43) and (46).} \quad (55)$$

Combining (53), (54), (55), we get

$$|\text{Error}_0| \leq C_{\#} S_{\min}(E_{\star}) \cdot \Lambda^{-2}. \quad (56)$$

Also, $C_{\#} \Lambda^{4\epsilon-2} |E_{\star}| \leq C'_{\#} \Lambda^{4\epsilon-2} S_{\min}(E_{\star})$, so (51), (52), (56) imply

$$\begin{aligned} \sum_{k \in \mathbb{Z} \cap [a, b]} E_k &= \int_a^b E(t) dt - E(b) \chi_{-}(b) - E(a) \chi_{+}(a) + \frac{1}{2} \frac{d}{dt} E(t) \Big|_{t=b} \tilde{\chi}(b) \\ &\quad - \frac{1}{2} \frac{d}{dt} E(t) \Big|_{t=a} \tilde{\chi}(a) + \text{Error}_1 \end{aligned} \quad (57)$$

with

$$|\text{Error}_1| \leq C_{\#} \Lambda^{4\epsilon-2} S_{\min}(E_{\star}). \quad (58)$$

Let us study in turn each of the terms on the right in (57). Regarding the integral in (57), we note that

$$\begin{aligned} \left| \int_a^b E(t) dt - \int_a^{b_0} E(t) dt \right| &\leq \sup\{|E(t)| : t \text{ between } b \text{ and } b_0\} \cdot |b - b_0| \\ &\leq \sup\{|E(t)| : |t - b_0| \leq \bar{C}_{\#} \Lambda^{4\epsilon-2}\} \cdot \bar{C}_{\#} \Lambda^{4\epsilon-2} \\ &\leq C_{\#} \sigma(b) \Lambda^{4\epsilon-2} \\ &\leq C_{\#} \Lambda^{4\epsilon-2} S(E_{\star}), \quad \text{by (45) and (46).} \end{aligned}$$

Thus,

$$\int_a^b E(t) dt = \int_a^{b_0} E(t) dt + \text{Error}_2, \quad \text{with } |\text{Error}_2| \leq C_{\#} \Lambda^{4\epsilon-2} S_{\min}(E_{\star}). \quad (59)$$

We wait until later to analyze $\int_a^{b_0} E(t) dt$.

Next, note that $|E(b)| = |E(b) - E(b_0)| \leq \sup\{|dE(t)/dt| : t \text{ between } b, b_0\} \cdot |b - b_0| \leq C_{\#} \sigma(b) \tau^{-1}(b) \Lambda^{4\epsilon-2}$ (by (40)) $\leq C_{\#} S_{\min}(E_{\star}) \cdot \Lambda^{4\epsilon-3}$, by (43), (46). Throwing away information, we conclude that

$$|E(b) \chi_{-}(b)| \leq C_{\#} \Lambda^{4\epsilon-2} S_{\min}(E_{\star}). \quad (60)$$

Next we study $-E(a)\chi_+(a)$. By definition of $E(t)$ and of a , we have $E(a) = E_*$, and $a = \Phi(E_*) = (1/\pi)\phi(E_*) + (1/48\pi)\psi(E_*) - \frac{1}{2}$. Our assumption on E_* was that $(1/\pi)\phi(E_*) = m + \xi$ with m an integer, and $|\xi| \leq A^{-2}$. Since $|\psi(E_*)| \leq C_\# A^{-1}$, we see that $a = m - \frac{1}{2} + (1/48\pi)\psi(E_*) + \xi$ lies near $m - \frac{1}{2}$. Hence $k = (\text{smallest integer} \geq a) = m$, and so by definition we have $\chi_+(a) = k - a - \frac{1}{2} = m - (m - \frac{1}{2} + (1/48\pi)\psi(E_*) + \xi) - \frac{1}{2} = -(1/48\pi)\psi(E_*) - \xi$. Therefore, $-E(a)\chi_+(a) = -E_* \cdot (-(1/48\pi)\psi(E_*) - \xi)$. Since $|E_* \cdot \xi| \leq C_\# A^{-2} S_{\min}(E_*)$ by (41), this may be rewritten as

$$-E(a)\chi_+(a) = +\frac{E_*}{48\pi}\psi(E_*) + \text{Error}_3,$$

$$\text{with } |\text{Error}_3| \leq C_\# A^{-2} S_{\min}(E_*).$$

Next we study the term $+\frac{1}{2}E'(b)\tilde{\chi}(b)$. Recall that $\tilde{\chi}(\cdot)$ is Lipschitz continuous, and that $b = (1/\pi)\phi(0) + (1/48\pi)\psi(0) - \frac{1}{2} + w_{hi}$, with $|w_{hi}| \leq C_\# A^{4\epsilon-2}$ and $|\psi(0)| \leq C_\# A^{-1}$. So

$$\left| \tilde{\chi}(b) - \tilde{\chi}\left(\frac{1}{\pi}\phi(0) - \frac{1}{2}\right) \right| \leq C_\# \left| \frac{1}{48\pi}\psi(0) + w_{hi} \right| \leq C_\# A^{-1}. \quad (62)$$

Also, (40), (42), (43), and $|b - b_0| \leq C_\# A^{4\epsilon-2}$ imply $|E'(b) - E'(b_0)| \leq C_\# |b - b_0| \sigma(b) \tau^{-2}(b) \leq C_\# A^{4\epsilon-2} \sigma(b) \tau^{-2}(b)$ and $|E'(b)| \leq C_\# \sigma(b) \tau^{-1}(b)$. From (43) and (46) we have therefore

$$|E'(b)| \leq C_\# A^{-1} S_{\min}(E_*), \quad (63)$$

and

$$|E'(b) - E'(b_0)| \leq C_\# A^{4\epsilon-4} S_{\min}(E_*). \quad (64)$$

Applying (62), (63), (64), we get

$$\begin{aligned} & \left| \frac{1}{2}E'(b)\tilde{\chi}(b) - \frac{1}{2}E'(b_0)\tilde{\chi}\left(\frac{1}{\pi}\phi(0) - \frac{1}{2}\right) \right| \\ & \leq \frac{1}{2}|E'(b)| \left| \tilde{\chi}(b) - \tilde{\chi}\left(\frac{1}{\pi}\phi(0) - \frac{1}{2}\right) \right| + \frac{1}{2}|E'(b) \\ & \quad - E'(b_0)| \left| \tilde{\chi}\left(\frac{1}{\pi}\phi(0) - \frac{1}{2}\right) \right| \\ & \leq C_\# A^{-1} S_{\min}(E_*) \cdot A^{-1} + C_\# A^{4\epsilon-2} S_{\min}(E_*) \leq C'_\# A^{-2} S_{\min}(E_*). \end{aligned} \quad (65)$$

By definition of $E(t)$, we have also $E'(t) = ((d/dE) \Phi(E)|_{E=E(t)})^{-1}$. Since $E(b_0) = 0$, this means

$$\begin{aligned} E'(b_0) &= \left(\frac{d}{dE} \Phi(E) \Big|_{E=0} \right)^{-1} = \left(\frac{1}{\pi} \phi'(0) + \frac{1}{48\pi} \psi'(0) \right)^{-1} \\ &= \pi(\phi'(0))^{-1} \left(1 + \frac{1}{48} \frac{\psi'(0)}{\phi'(0)} \right)^{-1}. \end{aligned}$$

From (6), (7), we get $|\psi'(0)/\phi'(0)| \leq C_{\#} A^{-2}$, so $|(1 + \frac{1}{48}(\psi'(0)/\phi'(0)))^{-1} - 1| \leq C_{\#} A^{-2}$. Hence, (66) implies

$$|E'(b_0) - \pi(\phi'(0))^{-1}| \leq C_{\#} A^{-2} |E'(b_0)| \leq C_{\#} A^{-2} \sigma(b) \tau^{-1}(b), \quad (67)$$

by (40), (42), (43) and the fact that $|b - b_0| \leq C_{\#} A^{4\epsilon - 2}$. Applying (43) and (46) to (67), we get $|E'(b_0) - \pi(\phi'(0))^{-1}| \leq C_{\#} A^{-3} S_{\min}(E_*)$. Combining this with (65), we conclude that

$$\begin{aligned} + \frac{1}{2} \frac{d}{dt} E(t) \Big|_{t=b} \cdot \tilde{\chi}(b) &= + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \text{Error}_4, \\ \text{with } |\text{Error}_4| &\leq C_{\#} A^{-2} S_{\min}(E_*). \end{aligned} \quad (68)$$

Next, we examine the term $-\frac{1}{2} E'(a) \tilde{\chi}(a)$. Since $E(a) = E_*$, we have (as in (66))

$$\begin{aligned} E'(a) &= \left(\frac{d\Phi}{dE}(E) \Big|_{E=E_*} \right)^{-1} = \left(\frac{1}{\pi} \phi'(E_*) + \frac{1}{48\pi} \psi'(E_*) \right)^{-1} \\ &= \pi(\phi'(E_*))^{-1} \cdot \left(1 + \frac{\psi'(E_*)}{48\phi'(E_*)} \right)^{-1}. \end{aligned} \quad (69)$$

Again, (6), (7) imply $|\psi'(E_*)/\phi'(E_*)| \leq C_{\#} A^{-2}$, so that (69) yields

$$|E'(a) - \pi(\phi'(E_*))^{-1}| \leq C_{\#} A^{-2} |E'(a)| \leq C'_{\#} A^{-3} S_{\min}(E_*), \quad (70)$$

since

$$|E'(a)| \leq C_{\#} \sigma(a) \tau^{-1}(a) \leq C_{\#} A^{-1} S_{\min}(E_*) \quad \text{by (40), (43), (46)}. \quad (71)$$

We turn to $\tilde{\chi}(a) = \tilde{\chi}((1/\pi) \phi(E_*) + (1/48) \psi(E_*) - \frac{1}{2})$, using again our assumption

$$\frac{1}{\pi} \phi(E_*) = m + \xi, \quad \text{with } m \text{ an integer and } |\xi| \leq A^{-2}.$$

We have therefore (by definition of $\tilde{\chi}$)

$$\begin{aligned}\tilde{\chi}(a) &= \tilde{\chi} \left(m - \frac{1}{2} + \frac{1}{48\pi} \psi(E_*) + \xi \right) \\ &= \inf_{k \in \mathbb{Z}} \left| \left(m - \frac{1}{2} + \frac{1}{48\pi} \psi(E_*) + \xi \right) - \left(k + \frac{1}{2} \right) \right|^2 - \frac{1}{12} \\ &= \left(\frac{1}{48\pi} \psi(E_*) + \xi \right)^2 - \frac{1}{12}, \quad \text{with } |\psi(E_*)| \leq C_{\#} A^{-1}.\end{aligned}$$

Hence

$$|\tilde{\chi}(a) + \tfrac{1}{12}| \leq C_{\#} A^{-2}. \quad (72)$$

Applying (70), (71), (72), we get

$$\begin{aligned}& \left| -\tfrac{1}{2} E'(a) \tilde{\chi}(a) - \left(-\tfrac{1}{2} \right) (\pi(\phi'(E_*))^{-1}) \cdot \left(-\tfrac{1}{12} \right) \right| \\ & \leq \tfrac{1}{2} |E'(a) - \pi(\phi'(E_*))^{-1}| \\ & \quad \cdot |\tilde{\chi}(a)| + \tfrac{1}{2} |\pi(\phi'(E_*))^{-1}| \cdot |\tilde{\chi}(a) + \tfrac{1}{12}| \\ & \leq C_{\#} A^{-3} S_{\min}(E_*) + C_{\#} A^{-1} S_{\min}(E_*) \cdot A^{-2} \\ & \leq C_{\#} A^{-3} S_{\min}(E_*).\end{aligned}$$

That is,

$$\begin{aligned}-\tfrac{1}{2} \frac{d}{dt} E(t) \Big|_{t=a} \cdot \tilde{\chi}(a) &= +\frac{\pi}{24} (\phi'(E_*))^{-1} + \text{Error}_5 \\ \text{with } |\text{Error}_5| &\leq C_{\#} A^{-3} S_{\min}(E_*).\end{aligned} \quad (73)$$

We can now substitute (59), (60), (61), (68), (73) into (57), to obtain the following result.

$$\begin{aligned}\sum_{k \in [a, b]} E_k &= \int_a^{b_0} E(t) dt + \frac{E_*}{48\pi} \psi(E_*) + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) \\ &\quad + \frac{\pi}{24} (\phi'(E_*))^{-1} + \text{Error}_6,\end{aligned} \quad (74)$$

with

$$|\text{Error}_6| \leq C_{\#} A^{4\epsilon-2} S_{\min}(E_*). \quad (75)$$

Next we calculate the integral in (74). Changing variable from t to $E = E(t)$ in that integral, we obtain

$$\begin{aligned}
 \int_a^{b_0} E(t) dt &= \int_{E_*}^0 E \frac{d\Phi}{dE} dE = \int_{E_*}^0 E \cdot \left(\frac{1}{\pi} \phi'(E) + \frac{1}{48\pi} \psi'(E) \right) dE \\
 &= \left(\frac{1}{\pi} \phi(E) + \frac{1}{48\pi} \psi(E) \right) \cdot E \Big|_{E_*}^0 \\
 &\quad - \int_{E_*}^0 \left(\frac{1}{\pi} \phi(E) + \frac{1}{48\pi} \psi(E) \right) dE \\
 &= -\frac{E_*}{\pi} \phi(E_*) - \frac{E_*}{48\pi} \psi(E_*) \\
 &\quad - \frac{1}{\pi} \int_{E_*}^0 \phi(E) dE - \frac{1}{48\pi} \int_{E_*}^0 \psi(E) dE.
 \end{aligned} \tag{76}$$

Thus, we have to integrate the phase functions ϕ and ψ . To integrate $\phi(E)$, we simply write

$$\begin{aligned}
 \int_{E_*}^0 \phi(E) dE &= \int_{\substack{E_* < E < 0 \\ V(x) < E}} (E - V(x))^{1/2} dx dE \\
 &= \int_{\substack{E < 0 \\ V(x) < E}} (E - V(x))^{1/2} dx dE \\
 &\quad - \int_{\substack{E < E_* \\ V(x) < E}} (E - V(x))^{1/2} dx dE \\
 &= \int \frac{2}{3} (-V(x))_+^{3/2} dx - \int \frac{2}{3} (E_* - V(x))_+^{3/2} dx,
 \end{aligned}$$

as follows from doing the dE -integration first. So

$$-\frac{1}{\pi} \int_{E_*}^0 dt \phi(E) dE = -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{2}{3\pi} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx. \tag{77}$$

To integrate $\psi(E)$, we use the following observation.

LEMMA 8. *The limit defining $\psi(E)$, namely*

$$\psi(E) = \lim_{\delta \rightarrow 0+} \left[\int_{E - V(x) > \delta} V''(x) \cdot (E - V(x))^{-3/2} dx - q(E) \delta^{-1/2} \right], \tag{78}$$

exists uniformly for $E \in [E_, 0]$. Also, $q(E)$ is continuous on $[E_*, 0]$.*

Proof. It is enough to establish uniform convergence and continuity of $q(E)$ for $E \in [E_*, 0]$ in a small neighborhood of a given $E_0 \in [E_*, 0]$. Because E_* is assumed to be strictly greater than the minimum of the potential, we can make a partition of unity $V''(x) = G_{\text{left}}(x) + G_{\text{center}}(x) + G_{\text{rt}}(x)$ in a neighborhood of $[x_{\text{left}}(E_0), x_{\text{rt}}(E_0)]$, with the following properties:

$$G_{\text{center}}(x) \text{ is } C^\infty \text{ and is supported in } \{E_0 - V(x) > \delta_0\} \text{ for some small, positive } \delta_0. \quad (79)$$

$$G_{\text{left}}(x) \text{ is } C^\infty \text{ and is supported in a small neighborhood of } x = x_{\text{left}}(E_0). \quad (80)$$

$$G_{\text{rt}}(x) \text{ is } C^\infty \text{ and is supported in a small neighborhood of } x = x_{\text{rt}}(E_0). \quad (81)$$

We write $\psi(E)$ in (78) as a sum of $\psi_{\text{left}}(E)$, $\psi_{\text{center}}(E)$, $\psi_{\text{rt}}(E)$ by replacing $V''(x)$ respectively by $G_{\text{left}}(x)$, $G_{\text{center}}(x)$, $G_{\text{rt}}(x)$. In place of $q(E)$ we have functions $q_{\text{left}}(E)$, $q_{\text{center}}(E)$, $q_{\text{rt}}(E)$. Evidently, for E near E_0 , the limit

$$\begin{aligned} \psi_{\text{center}}(E) = \lim_{\delta \rightarrow 0+} \left[\int_{E - V(x) > \delta} G_{\text{center}}(x) (E - V(x))^{-3/2} dx \right. \\ \left. - q_{\text{center}}(E) \delta^{-1/2} \right] \end{aligned}$$

exists uniformly, and $q_{\text{center}}(E) \equiv 0$. The discussions for $\psi_{\text{left}}(E)$ and $\psi_{\text{rt}}(E)$ are analogous, so we just study

$$\psi_{\text{left}}(E) = \lim_{\delta \rightarrow 0+} \left[\int_{E - V(x) > \delta} G_{\text{left}}(x) \cdot (E - V(x))^{-3/2} dx - q_{\text{left}}(E) \delta^{-1/2} \right]. \quad (82)$$

For (x, E) near $(x_{\text{left}}(E_0), E_0)$, we make the smooth change of variable $(x, E) \rightarrow (\xi, E)$, $\xi = E - V(x)$. (Since $-V'(x) > 0$ at $x = x_{\text{left}}(E_0)$, this is a smooth change of variables. Here we make crucial use of our assumption $E_* > V(x_0) = \min V$.)

The change of variable is well-defined for $x \in \text{supp } G_{\text{left}}$ and E near E_0 and produces $|\xi| \leq A$ for such (x, E) . Here A is some positive number independent of x, E . Therefore, using ξ in place of x as the independent variable in (82), we get

$$\psi_{\text{left}}(E) = \lim_{\delta \rightarrow 0+} \left[\int_{\delta}^A \theta(\xi, E) \cdot \xi^{-3/2} d\xi - q_{\text{left}}(E) \delta^{-1/2} \right] \quad (83)$$

for a smooth function $\theta(\xi, E)$. Writing

$$\begin{aligned} \int_{\delta}^A \theta(\xi, E) \cdot \xi^{-3/2} d\xi &= \int_{\delta}^A \left[\frac{\theta(\xi, E) - \theta(0, E)}{\xi} \right] \xi^{-1/2} d\xi \\ &\quad + \theta(0, E) \int_{\delta}^A \xi^{-3/2} d\xi, \end{aligned}$$

we see that the limit in (83) exists uniformly, and that $q_{\text{left}}(E) = 2\theta(0, E)$ is continuous. Since the quantity in brackets in (83) is equal to that in (82) the proof is complete. ■

Lemma 8 gives

$$\begin{aligned} \int_{E_*}^0 \psi(E) dE &= \lim_{\delta \rightarrow 0+} \left[\int_{\substack{E_* < E < 0 \\ E - V(x) > \delta}} \int V''(x) \cdot (E - V(x))^{-3/2} dx dE \right. \\ &\quad \left. - \int_{E_*}^0 q(E) dE \cdot \delta^{-1/2} \right]. \end{aligned} \quad (84)$$

In particular, the limit in (84) exists. We have

$$\begin{aligned} &\int_{\substack{E_* < E < 0 \\ E - V(x) > \delta}} \int V''(x) \cdot (E - V(x))^{-3/2} dx dE \\ &= \int_{\substack{E < 0 \\ E - V(x) > \delta}} \int V''(x) \cdot (E - V(x))^{-3/2} dx dE \\ &\quad - \int_{\substack{E < E_* \\ E - V(x) > \delta}} \int V''(x) \cdot (E - V(x))^{-3/2} dx dE \\ &= \int_{V(x) < -\delta} V''(x) \left[\int_{V(x) + \delta}^0 (E - V(x))^{-3/2} dE \right] dx \\ &\quad - \int_{V(x) < E_* - \delta} V''(x) \left[\int_{V(x) + \delta}^{E_*} (E - V(x))^{-3/2} dE \right] dx \\ &= -2 \int_{V(x) < -\delta} V''(x) \cdot (-V(x))^{-1/2} dx + 2\delta^{-1/2} \int_{V(x) < -\delta} V''(x) dx \\ &\quad + 2 \int_{V(x) < E_* - \delta} V''(x) \cdot (E_* - V(x))^{-1/2} dx \\ &\quad - 2\delta^{-1/2} \int_{V(x) < E_* - \delta} V''(x) dx. \end{aligned}$$

Putting this into (84), we find that

$$\begin{aligned} \int_{E_*}^0 \psi(E) dE = & \lim_{\delta \rightarrow 0+} \left[-2 \int_{V(x) < -\delta} V''(x) \cdot (-V(x))^{-1/2} dx \right. \\ & + 2 \int_{V(x) < E_* - \delta} V''(x) \cdot (E_* - V(x))^{-1/2} dx \\ & \left. + \delta^{-1/2} \left\{ 2 \int_{E_* - \delta < V(x) < -\delta} V''(x) dx - \int_{E_*}^0 q(E) dE \right\} \right]. \end{aligned} \quad (85)$$

In particular, the limit in (85) exists. As $\delta \rightarrow 0+$, we have

$$\begin{aligned} & \int_{V(x) < -\delta} V''(x) \cdot (-V(x))^{-1/2} dx \\ & \rightarrow \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ & \int_{V(x) < E_* - \delta} V''(x) \cdot (E_* - V(x))^{-1/2} dx \\ & \rightarrow \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx \\ & \int_{E_* - \delta < V(x) < -\delta} V''(x) dx \\ & = \int_{E_* < V(x) < 0} V''(x) dx + O(\delta). \end{aligned}$$

Hence, the limit in (85) can exist only when

$$\left\{ 2 \int_{E_* < V(x) < 0} V''(x) dx - \int_{E_*}^0 q(E) dE \right\} = 0; \quad (86)$$

and when (86) holds, then (85) simplifies to

$$\begin{aligned} \int_{E_*}^0 \psi(E) dE = & -2 \int_{I_{\text{BVP}}} V''(x) (-V(x))_+^{-1/2} dx \\ & + 2 \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx. \end{aligned} \quad (87)$$

Putting (77) and (87) into (76), we find that

$$\begin{aligned} \int_a^{b_0} E(t) dt &= -\frac{E_*}{\pi} \phi(E_*) - \frac{E_*}{48\pi} \psi(E_*) \\ &\quad - \frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{2}{3\pi} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx \\ &\quad + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ &\quad - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx. \end{aligned}$$

Substituting this into (74) and noting the cancellation of the terms $E_* \psi(E_*)/48\pi$, we get

$$\begin{aligned} \sum_{k \in [a, b]} E_k &= -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{2}{3\pi} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx \\ &\quad + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ &\quad - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx \\ &\quad - \frac{E_*}{\pi} \phi(E_*) + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \frac{\pi}{24} (\phi'(E_*))^{-1} \\ &\quad + \text{Error}_6, \quad \text{with Error}_6 \text{ estimated by (75)}. \end{aligned} \quad (88)$$

The corollary to Lemma 7 and the definition (44) of a, b show that the left-hand side of (88) is the sum of the eigenvalues of $-d^2/dx^2 + V(x)$ belonging to $[E_*, 0]$.

We record our results (88), (75) in the following lemma.

LEMMA ON TRUNCATED EIGENVALUE SUMS. *Suppose we are given $\varepsilon, K, N, \hat{c}, x_0 \in I \subset I_{\text{BVP}} \subset \mathbb{R}^1$, $S(x)$, $B(x)$, $V(x)$ satisfying hypotheses (Z0)–(Z9) of the WKB Density Theorem. Let $E_* \in (V(x_0), 0]$, and assume that $\phi(E_*) < \phi(0) - 1$, and that $(1/\pi)\phi(E_*)$ differs from an integer by at most $(1/\pi)A^{-2}$. Let $H = -d^2/dx^2 + V(x)$ on I_{BVP} , with Dirichlet or*

Neumann boundary conditions. Let X denote the sum of the eigenvalues of H belonging to the interval $[E_, 0]$. Then*

$$\begin{aligned} X = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ & + \frac{2}{3\pi} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx \\ & + \frac{\pi}{24} (\phi'(E_*))^{-1} - \frac{E_*}{\pi} \phi(E_*) + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) \\ & + \text{Error}, \end{aligned}$$

with

$$|\text{Error}| \leq C_* A^{4\varepsilon-2} \cdot \inf_{\substack{x \in I \\ V(x) < E_*}} S(s).$$

The constant C_* depends only on $\varepsilon, K, N, c, C, c_1, C_\alpha, \hat{c}$ in hypotheses (Z0)–(Z9).

THE FIRST WKB EIGENVALUE SUM THEOREM

Our setting is as in the WKB Density Theorem, and we assume the hypotheses (Z0)–(Z9) of that result. Thus, $H = -d^2/dx^2 + V(x)$ on I_{BVP} , with Dirichlet or Neumann boundary conditions. Let $\text{sneg}(H)$ denote the sum of the negative eigenvalues of H .

FIRST WKB EIGENVALUE SUM THEOREM.

$$\begin{aligned} \text{sneg}(H) = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ & + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \text{Error}, \end{aligned}$$

$$\text{with } |\text{Error}| \leq A^{5\varepsilon-2} |V(x_0)|.$$

Recall that $V(x)$ attains its minimum at $x = x_0$, and that $\phi(0) = \int_{I_{\text{BVP}}} (-V(x))_+^{1/2} dx$, $\phi'(0) = \frac{1}{2} \int_{I_{\text{BVP}}} (-V(x))_+^{-1/2} dx$, while $\tilde{\chi}(t) = \min_{k \in \mathbb{Z}} |t - k - \frac{1}{2}|^2 - \frac{1}{12}$.

Proof. Take $E_* = V(x_0) + \tau^2$ with τ very small, apply the Lemma on Truncated Eigenvalue Sums, and let $\tau \rightarrow 0$. The hypotheses imposed on E_* in that lemma are $E_* \in (V(x_0), 0]$, $|(1/\pi)\phi(E_*) - \text{integer}| < (1/\pi)A^{-2}$ and $\phi(E_*) < \phi(0) - 1$. These hypotheses hold trivially, for $\tau \neq 0$ small enough. In fact, $(1/\pi)\phi(E_*) \rightarrow 0$ as $\tau \rightarrow 0$, so the hypotheses on E_* follow, once we check that $\phi(0) > 2$. However, contained in the hypotheses (Z0)–(Z9) is the fact that $V(x) < -c_* S(x_0)$ for $|x - x_0| < c_* B(x_0)$. (Here we use c_* , C_* as in the previous section, to denote constants depending only on ε , K , N , c , C , c_1 , C_2 , \hat{c} in the hypotheses (Z0)–(Z9).) Therefore

$$\begin{aligned}\phi(0) &= \int_{I_{\text{BVP}}} (-V(x))_+^{1/2} dx \geq \int_{|x-x_0| < c_* B(x_0)} c_* S^{1/2}(x_0) dx \\ &= c'_* S^{1/2}(x_0) B(x_0) \\ &= c'_* \lambda(x_0) \geq c'_* A > 2,\end{aligned}$$

as needed. So the Lemma on Truncated Eigenvalue Sums holds for

$$E_* = V(x_0) + \tau^2, \quad \tau \text{ small but nonzero.} \quad (1)$$

Since all the eigenvalues of $-d^2/dx^2 + V(x)$ are greater than $V(x_0) + \delta_0$ for some small, fixed δ_0 , it follows that $X = \text{sum of eigenvalues of } H \text{ belonging to } [E_*, 0] = \text{sneg}(H)$ for $\tau \neq 0$ small enough. Hence the Lemma on Truncated Eigenvalues Sums gives, for small non-zero τ ,

$$\begin{aligned}\text{sneg}(H) &= -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx \\ &\quad + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ &\quad + \frac{2}{3\pi} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx \\ &\quad - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx \\ &\quad + \frac{\pi}{24} (\phi'(E_*))^{-1} - \frac{E_*}{\pi} \phi(E_*) \\ &\quad + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \text{Error}(\tau),\end{aligned} \quad (2)$$

with

$$|\text{Error}(\tau)| \leq C_{\#} A^{4\epsilon-2} \cdot \inf_{\substack{x \in I \\ V(x) < V(x_0) + \tau^2}} S(x). \quad (3)$$

We study what happens to the various terms on the right-hand side of (2) when $\tau \rightarrow 0$. The first two integrals and the term involving $\tilde{\chi}$ are of course independent of τ . Also, $-(E_{\#}/\pi) \phi(E_{\#})$ and $\int_{I_{\text{BVP}}} (E_{\#} - V(x))_+^{3/2} dx$ evidently tend to zero as $E_{\#} \rightarrow V(x_0) +$, i.e., as $\tau \rightarrow 0$. The terms that require a little work are $(\pi/24)(\phi'(E_{\#}))^{-1}$ and $-(1/24\pi) \int_{I_{\text{BVP}}} V''(x) \cdot (E_{\#} - V(x))_+^{-1/2} dx$. We shall see that

$$\lim_{\tau \rightarrow 0} \left\{ \frac{\pi}{24} (\phi'(E_{\#}))^{-1} - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_{\#} - V(x))_+^{-1/2} dx \right\} = 0. \quad (4)$$

Assuming (4) for a moment, we can pass to the limit in (2), to derive the equations

$$\begin{aligned} \text{sneg}(H) = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx \\ & + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ & + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \text{Error}, \end{aligned} \quad (5)$$

with

$$|\text{Error}| \leq C_{\#} A^{4\epsilon-2} \cdot \limsup_{E_{\#} \rightarrow V(x_0) +} \left\{ \inf_{\substack{x \in I \\ V(x) < E_{\#}}} S(x) \right\}. \quad (6)$$

The right-hand side of (6) is dominated by $C_{\#} A^{4\epsilon-2} S(x_0)$ since $x_0 \in I$, $V(x_0) < E_{\#}$. Since the inequalities $V(x_0) < -c_{\#} S(x_0) < 0$ are contained in the hypotheses (Z0)–(Z9), it follows that $S(x_0) \leq C_{\#} |V(x_0)|$. Therefore, (6) implies

$$|\text{Error}| \leq C'_{\#} A^{4\epsilon-2} |V(x_0)| \leq A^{5\epsilon-2} |V(x_0)|. \quad (7)$$

Equations (5), (7) are the conclusions of the First WKB Eigenvalue Sum Theorem. Thus, our theorem is reduced to proving (4).

To prove (4), it is convenient to make a change of variable, which we prepare to define. Since $V'(x_0) = 0$ Taylor's theorem gives

$$V(x) = V(x_0) + h(x) \cdot (x - x_0)^2 \quad (8)$$

for a smooth function $h(x)$ defined in a neighborhood of x_0 . Differentiating (8) twice, we find that $V''(x_0) = 2h(x_0)$. Hence, $h(x_0) > 0$, so h has a smooth square root $f(x)$ defined in a neighborhood of x_0 , with $f(x_0) = (\frac{1}{2}V''(x_0))^{1/2}$. Setting $\xi(x) = f(x) \cdot (x - x_0)$, we obtain

$$V(x) = V(x_0) + (\xi(x))^2 \quad (9)$$

from (8), and

$$\xi(x_0) = 0, \quad \xi'(x_0) = f(x_0) = (\tfrac{1}{2}V''(x_0))^{1/2} > 0 \quad (10)$$

from the definition of $\xi(x)$. Here, $\xi(x)$ is smooth in a neighborhood of x_0 . Our change of variable is $s = \xi(x)$, which is inverted by a function $x = y(s)$, satisfying

$$y(0) = x_0, \quad y'(0) = (\tfrac{1}{2}V''(x_0))^{-1/2} > 0. \quad (11)$$

The function $y(s)$ is smooth in a neighborhood of the origin. Changing variable from x to $s = \xi(x)$ gives $E_* - V(x) = \tau^2 - s^2$ by (1), (9), so that

$$\begin{aligned} \int_{I_{\text{BVP}}} V''(x)(E_* - V(x))_+^{-1/2} dx \\ = \int_{-\infty}^{\infty} V''(y(s)) \cdot (\tau^2 - s^2)_+^{-1/2} y'(s) ds \end{aligned} \quad (12)$$

and

$$\phi'(E_*) = \frac{1}{2} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{-1/2} dx = \frac{1}{2} \int_{-\infty}^{\infty} (\tau^2 - s^2)_+^{-1/2} y'(s) ds. \quad (13)$$

For a general function $\theta(s)$ smooth in a neighborhood of the origin, and for small, nonzero τ we note that

$$\begin{aligned} \int_{-\infty}^{\infty} (\tau^2 - s^2)_+^{-1/2} \theta(s) ds = \theta(0) \cdot \left[\int_{-\infty}^{\infty} (\tau^2 - s^2)_+^{-1/2} ds \right] \\ + \left[\int_{-\infty}^{\infty} (\tau^2 - s^2)_+^{-1/2} \{\theta(s) - \theta(0)\} ds \right]. \end{aligned}$$

The first integral on the right is identically equal to π . The second integral on the right is dominated by $C \int_{-\infty}^{\infty} |s|(\tau^2 - s^2)_+^{-1/2} ds \leq C|\tau| \int_{-\infty}^{\infty} (\tau^2 - s^2)_+^{-1/2} ds = C\pi|\tau|$. It follows that $\lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} \theta(s) \cdot (\tau^2 - s^2)_+^{-1/2} ds = \pi\theta(0)$ whenever $\theta(s)$ is smooth near the origin. Taking $\theta(s) = V''(y(s)) \cdot y'(s)$ and $\theta(s) = y'(s)$, we see from (12), (13) that

$\lim_{\tau \rightarrow 0} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx = \pi V''(y(0)) y'(0)$ and $\lim_{\tau \rightarrow 0} \phi'(E_*) = (\pi/2) y'(0)$. Substituting (11), we get

$$\begin{aligned} \lim_{\tau \rightarrow 0} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx \\ = \pi V''(x_0) \cdot \left(\frac{1}{2} V''(x_0) \right)^{-1/2} = \pi \sqrt{2} (V''(x_0))^{+1/2} \end{aligned} \quad (14)$$

and

$$\lim_{\tau \rightarrow 0} \phi'(E_*) = \frac{\pi}{2} \left(\frac{1}{2} V''(x_0) \right)^{-1/2} = \frac{\pi}{\sqrt{2}} (V''(x_0))^{-1/2}. \quad (15)$$

Hence, (4) is reduced to checking that

$$\frac{\pi}{24} \left[\frac{\pi}{\sqrt{2}} (V''(x_0))^{-1/2} \right]^{-1} - \frac{1}{24\pi} \left[\pi \sqrt{2} (V''(x_0))^{+1/2} \right] = 0,$$

which is correct.

The proof of the First WKB Eigenvalue Theorem is complete. ■

LOW EIGENVALUES IN A POTENTIAL WITH A COULOMB SINGULARITY

In this section and the next, we prove a variant of the First WKB Theorem on Eigenvalue Sums, which applies to potentials that look like

$$V_c(x) = \frac{l(l+1)}{x^2} + E_0 - \frac{Z}{x} \quad (1)$$

near $x=0$. Here, we show that if $V(x)$ is given exactly by (1) near the origin, then the low eigenvalues of $-d^2/dx^2 + V(x)$ on $(0, \infty)$ are very close to the familiar eigenvalues of $-d^2/dx^2 + V_c(x)$. In the next section, we combine this information with the Lemma on Truncated Eigenvalue Sums, to compare the eigenvalue sum for $-d^2/dx^2 + V(x)$ with that for $-d^2/dx^2 + V_c(x)$. We begin with a slight variant of the Agmon lemma from [FS2].

LEMMA 1. Suppose $-d^2/dx^2 u + Vu = Eu$ on $(0, \infty)$, with $\|u\|_{L^2(0, \infty)} = 1$. Suppose also that V is smooth on $(0, \infty)$, and that

$$V(x) - E \geq \frac{1}{2} |E_*| \quad \text{for } x > x_*.$$

Then

$$\|u\|_{L^2(2x_*, \infty)}^2 \leq C_N (x_*^2 |E_*|)^{-N} \{1 + x_*^2 \sup_{x_*/8 < x < x_*} |V(x) - E|\}$$

and

$$\|u'\|_{L^2(2x_*, \infty)}^2 \leq C_N x_*^{-2} (x_*^2 |E_*|)^{-N} \{1 + x_*^2 \sup_{x_*/8 < x < x_*} |V(x) - E|\}$$

with C_N depending only on N .

Proof. We may assume u is real. For $y > 0$ we have $Av_{[y/2, y]} |u|^2 = (2/y) \|u\|_{L^2(y/2, y)}^2 \leq 2/y$. Putting $y/4$ for y gives also $Av_{[y/8, y/4]} |u|^2 \leq 8/y$. Hence we can find $x_1 \in (y/8, y/4)$ and $x_2 \in (y/2, y)$ with $|u(x_1)|^2 \leq 8/y$, $|u(x_2)|^2 \leq 2/y$. The mean-value theorem produces an $x \in [x_1, x_2] \subset (y/8, y)$ with $2u'(x)u(x) = (u^2(x_2) - u^2(x_1))/(x_2 - x_1)$, hence

$$|u'(x)u(x)| \leq \frac{1}{2} \frac{8/y + 2/y}{y/4} = \frac{20}{y^2}.$$

Putting $y = x_*$, we obtain an $x_0 \in (x_*/8, x_*)$ with

$$|u'(x_0)u(x_0)| \leq 20x_*^{-2}. \quad (2)$$

Instead letting $y \rightarrow \infty$, we obtain a sequence $x_v \rightarrow \infty$, with

$$u'(x_v)u(x_v) \rightarrow 0. \quad (3)$$

Now suppose $\phi \in C^\infty(0, \infty)$. Integration by parts yields

$$\begin{aligned} 0 &= \int_{x_0}^{x_v} e^\phi u \left\{ -\frac{d^2}{dx^2} u + (V - E) u \right\} dx = -e^\phi uu' \Big|_{x_0}^{x_v} \\ &\quad + \int_{x_0}^{x_v} e^\phi \{ (u')^2 + (V - E) u^2 + \phi' uu' \} dx \\ &\geq -e^\phi uu' \Big|_{x_0}^{x_v} + \int_{x_0}^{x_v} e^\phi \left\{ \frac{1}{2} (u')^2 + (V - E - 100(\phi')^2) u^2 \right\} dx \end{aligned} \quad (4)$$

since $\phi' uu' \geq -\frac{1}{2}(u')^2 - 100(\phi')^2 u^2$.

We want to take $\phi(x) = 0$ for $x < x_*$, $\phi(x) = 10^{-3} |E_*|^{1/2} (x - x_*)$ for $x_* \leq x \leq y$, $\phi(x) = 10^{-3} |E_*|^{1/2} (y - x_*)$ for $y \leq x$. Here y is a given point with $2x_* < y < x_v$.

Although ϕ is not C^∞ , we can find $\phi_\varepsilon \in C^\infty$ with $\phi_\varepsilon(x) \rightarrow \phi(x)$ and $\phi'_\varepsilon(x) \rightarrow \phi'(x)$ for $x \neq x_*, y$; and with $|\phi_\varepsilon|, |\phi'_\varepsilon|$ bounded uniformly in ε . Applying (4) to ϕ_ε and passing to the limit, we obtain (4) for the desired ϕ .

For $x < x_*$ we have $\phi = 0$, $\phi' = 0$.

For $x_* < x$ we have $V - E - 100(\phi')^2 \geq V - E - 10^{-4} |E_*| > \frac{1}{4} |E_*| > 0$.
Hence (4) implies

$$\begin{aligned} & |u(x_0) u'(x_0)| + |e^{\phi(x_v)} u(x_v) u'(x_v)| + \int_{x_0}^{x_*} |V(x) - E| \cdot |u(x)|^2 dx \\ & \geq \int_{x_*}^{x_v} e^{\phi} \left\{ \frac{1}{2} (u')^2 + \frac{1}{4} |E_*| (u(x))^2 \right\} dx. \end{aligned}$$

Note that $\phi(x_v) = \phi(y)$ since $x_v \geq y$, and that

$$\begin{aligned} \int_{x_0}^{x_*} |V(x) - E| |u(x)|^2 dx & \leq \sup_{x \in (x_0, x_*)} |V(x) - E| \cdot \|u\|^2 \\ & \leq \sup_{x_*/8 < x < x_*} |V(x) - E|. \end{aligned}$$

Hence

$$\begin{aligned} & |u(x_0) u'(x_0)| + |e^{\phi(y)} u(x_v) u'(x_v)| + \sup_{x_*/8 < x < x_*} |V(x) - E| \\ & \geq \int_{x_*}^{x_v} e^{\phi} \left\{ \frac{1}{2} (u')^2 + \frac{1}{4} |E_*| u^2 \right\} dx. \end{aligned}$$

The expression in curly brackets is non-negative everywhere, and $\phi(x) \geq 10^{-3} |E_*|^{1/2} x_*$ for $x \in [2x_*, y] \subset [x_*, x_v]$. Hence,

$$\begin{aligned} & |u(x_0) u'(x_0)| + |e^{\phi(y)} u(x_v) u'(x_v)| + \sup_{x_*/8 < x < x_*} |V(x) - E| \\ & \geq \int_{2x_*}^y \left\{ \frac{1}{2} (u')^2 + \frac{1}{4} |E_*| (u^2) \right\} dx \cdot \exp(10^{-3} |E_*|^{1/2} x_*). \end{aligned} \quad (5)$$

In (5) we fix $y > 2x_*$ and let $v \rightarrow \infty$. From (2), (3), (5) we obtain

$$\begin{aligned} & \exp(-10^{-3} |E_*|^{1/2} x_*) \cdot \{20x_*^{-2} + \sup_{x_*/8 < x < x_*} |V(x) - E|\} \\ & \geq \int_{2x_*}^y \left\{ \frac{1}{2} (u')^2 + \frac{1}{4} |E_*| (u^2) \right\} dx. \end{aligned}$$

This holds for any $y > 2x_*$, so we can let $y \rightarrow \infty$ to get

$$\begin{aligned} \int_{2x_*}^{\infty} (u')^2 dx & \leq 40x_*^{-2} \{1 + x_*^2 \sup_{x_*/8 < x < x_*} |V(x) - E|\} \\ & \cdot \exp(-10^{-3} |E_*|^{1/2} x_*) \end{aligned}$$

$$\int_{2x_*}^{\infty} (u)^2 dx \leq 80(|E_*| x_*^2)^{-1} \{1 + x_*^2 \sup_{x_*/8 < x < x_*} |V(x) - E|\} \\ \cdot \exp(-10^{-3} |E_*|^{1/2} x_*).$$

These estimates are much stronger than the conclusion of the lemma. ■

LEMMA 2. Suppose $V_1(x)$ and $V_2(x)$ are smooth potentials on $(0, \infty)$, with $V_1(x) = V_2(x)$ for $x < 10x_*$. Assume also $V_1(x), V_2(x) \geq \frac{1}{2}E_*$ for $x > x_*$, with a given $E_* < 0$. Define $H_1 = -d^2/dx^2 + V_1(x)$ and $H_2 = -d^2/dx^2 + V_2(x)$ on $(0, \infty)$ with Dirichlet boundary conditions. Let $E_1 < E_*$ be an eigenvalue of H_1 . Assume also that $\sup_{x_*/8 < x < x_*} |V_1(x) - E_1| \leq (|E_*| x_*^2)^K |E_*|$, and that $|E_*| x_*^2$ is greater than a certain large, positive number depending only on K, N . Then there is a point E_2 in the spectrum of H_2 , with

$$|E_1 - E_2| \leq |E_*| \cdot (|E_*| x_*^2)^{-N}.$$

Proof. Let $u(x)$ be the (normalized) eigenfunction corresponding to E_1 . For $x > x_*$ we have $V_1(x) - E_1 \geq \frac{1}{2}E_* - E_* = \frac{1}{2}|E_*|$, so Lemma 1 applies. In view of our estimate on $\sup_{x_*/8 < x < x_*} |V_1(x) - E_1|$, we get, from Lemma 1,

$$\|u\|_{L^2(2x_*, \infty)}^2 \leq C_{N_1} (x_*^2 |E_*|)^{K+1-N_1}, \quad (6)$$

and

$$\|u'\|_{L^2(2x_*, \infty)}^2 \leq C_{N_1} x_*^{-2} (x_*^2 |E_*|)^{K+1-N_1} \quad (7)$$

for N_1 to be picked, depending on K and N .

Take $\chi(x)$ satisfying $\chi(x) = 1$ for $x \leq 2x_*$, $\chi(x) = 0$ for $x \geq 10x_*$, $0 \leq \chi \leq 1$ everywhere, $|(d/dx)^m \chi(x)| \leq C_m x_*^{-m}$. Then set $w = \chi u$.

Note that $\|w\| \geq \|u\| - \|(1 - \chi)u\| = 1 - \|(1 - \chi)u\| \geq 1 - \|u\|_{L^2(2x_*, \infty)} > \frac{1}{2}$ by (6), if we take N_1 large enough. Also,

$$\begin{aligned} (H_2 - E_1)w &= \left(-\frac{d^2}{dx^2} + V_2 - E_1\right)(\chi u) \\ &= \chi \left(-\frac{d^2}{dx^2} + V_2 - E_1\right)u - 2\chi'u' - \chi''u \\ &= -2\chi'u' - \chi''u, \end{aligned}$$

since $V_2 = V_1$ in $\text{supp } \chi$, and $(H_1 - E_1)u = 0$. Since χ', χ'' are supported in $\{x > 2x_*\}$ and $|\chi'(x)| \leq Cx_*^{-1}$, $|\chi''(x)| \leq Cx_*^{-2}$, it follows from (6), (7), (8) that

$$\begin{aligned} \|(H_2 - E_1)w\|^2 &\leq Cx_*^{-2} \|u'\|_{L^2(2x_*, \infty)}^2 + Cx_*^{-4} \|u\|_{L^2(2x_*, \infty)}^2 \\ &\leq C_{N_1} x_*^{-4} (x_*^2 |E_*|)^{K+1-N_1} = C_{N_1} |E_*|^2 (x_*^2 |E_*|)^{K-1-N_1}. \end{aligned}$$

Taking $N_1 > K - 1 + 2N$ and assuming $x_*^2 |E_*|$ large enough, we conclude that

$$\|(H_2 - E_1)w\| \leq \frac{1}{2} |E_*| (x_*^2 |E_*|)^{-N}. \quad (9)$$

Since $\|w\| > \frac{1}{2}$, it follows from spectral theory and (9) that $|E_1 - E_2| \leq |E_*| (x_*^2 |E_*|)^{-N}$ for some E_2 in the spectrum of H_2 . This is the conclusion of the lemma. ■

COROLLARY. *Under the assumptions of Lemma 2, let u be the normalized eigenfunction of H_1 corresponding to E_1 . Then there is a function w on $(0, \infty)$, satisfying $\|u - w\| \leq (|E_*| x_*^2)^{-N}$ and $\|(H_2 - E_1)w\| \leq |E_*| \cdot (x_*^2 |E_*|)^{-N}$.*

Proof. The second estimate is (9) and the first is immediate from (6), provided we take N_1 large enough and use the function w constructed in the proof of Lemma 2. ■

LEMMA ON LOW EIGENVALUES IN COULOMB SINGULARITIES. *Suppose we are given E_0, E_*, Z, l, x_* satisfying the following*

- (a) $cZ^{4/3} < E_0 < CZ^{4/3}$
- (b) $-Z^2 < E_* < -Z^{4/3}$
- (c) $\min_{n \in \mathbb{Z}} |E_* - (E_0 - Z^2/4n^2)| > Z^{-500}$
- (d) l is a non-negative integer, and $l < CZ^{1/3}$
- (e) $|E_*| x_*^2 > Z^{1/100}$ and $x_* > Z^{-1}$

(f) Z is greater than a large positive number determined by c and C in (a) and (d).

Let $V(x)$ be a smooth potential in $(0, \infty)$, equal to $V_c(x) \equiv l(l+1)/x^2 + E_0 - Z/x$ for $x \leq 10x_*$. Assume also that $V(x), V_c(x) > \frac{1}{2}E_*$ for $x > x_*$. Let $H = -d^2/dx^2 + V(x)$ on $(0, \infty)$, with Dirichlet boundary conditions. Then the spectrum of H in $(-\infty, E_*)$ consists of eigenvalues $E_0 - Z^2/4n^2 + \text{Error}_n$ for all integers n in the range $l < n < Z/2(E_0 - E_*)^{1/2}$, and with $|\text{Error}_n| \leq Z^{-500}$.

Proof. We start with a few preliminary remarks. For $x > x_*$ we have $V(x) \geq \frac{1}{2}E_* > E_*$, and therefore the spectrum of H in $(-\infty, E_*)$ consists at most of finitely many eigenvalues. Also, note that $V(x) \geq \frac{1}{2}E_* \geq -\frac{1}{2}Z^2$ for $x > x_*$, while $V(x) = V_c(x) \geq -Z/x$ for $x \leq x_*$. Hence $V(x) \geq -Z/x - \frac{1}{2}Z^2$ for all $x \in (0, \infty)$, which implies that the lowest eigenvalue of H is greater than or equal to that of $-d^2/dx^2 - Z/x - \frac{1}{2}Z^2$, which is $-\frac{3}{4}Z^2$.

If $E < 0$ is an eigenvalue of either H or $H_c = -d^2/dx^2 + V_c(x)$, then $-\frac{3}{4}Z^2 \leq E < 0$, and

$$\begin{aligned} \sup_{x_*/8 < x < x_*} |V(x) - E| &= \sup_{x_*/8 < x < x_*} |V_c(x) - E| \\ &\leq \sup_{x_*/8 < x < x_*} \left(\frac{l(l+1)}{x^2} + E_0 + \frac{Z}{x} \right) + |E| \\ &\leq CZ^{2/3}x_*^{-2} + CZ^{4/3} + CZx_*^{-1} + \frac{3}{4}Z^2 \leq CZ^{8/3}, \end{aligned}$$

whereas

$$(|E_*| x_*^2)^K |E_*| \geq Z^{K/100} \cdot Z^{4/3} \gg CZ^{8/3} \quad \text{for large } K.$$

Hence

$$\sup_{x_*/8 < x < x_*} |V(x) - E| = \sup_{x_*/8 < x < x_*} |V_c(x) - E| \leq (|E_*| x_*^2)^K |E_*|, \quad (10)$$

as in the hypotheses of Lemma 2.

We will prove three statements:

Every eigenvalue $E < E_*$ of H may be written in the form $E = E_0 - Z^2/4n^2 + \text{Error}$, where $|\text{Error}| < Z^{-500}$ and n is an integer satisfying $l < n < Z/2(E_0 - E_*)^{1/2}$. (11)

If $E, \tilde{E} < E_*$ are eigenvalues of H , and if for an integer n satisfying $l < n < Z/2(E_0 - E_*)^{1/2}$ we have $E = E_0 - Z^2/4n^2 + \text{Error}$ and $\tilde{E} = E_0 - Z^2/4n^2 + \widetilde{\text{Error}}$, with $|\text{Error}|, |\widetilde{\text{Error}}| < Z^{-500}$, then $E = \tilde{E}$. (12)

Given an integer n with $l < n < Z/2(E_0 - E_*)^{1/2}$, we can find an eigenvalue $E < E_*$ of H , which can be written as $E = E_0 - Z^2/4n^2 + \text{Error}$ with $|\text{Error}| < Z^{-500}$. (13)

Together these statements make up the conclusion of the lemma. First we prove (11). We take $H_1 = H$, $E_1 = E$, $H_2 = H_c = -d^2/dx^2 + V_c$ and apply Lemma 2. (Note that the hypotheses of Lemma 2 hold, by virtue of

(10).) Thus there is an E_2 in the spectrum of H_c , with $|E - E_2| \leq |E_*| (x_*^2 |E_*|)^{-N} \leq Z^2 \cdot Z^{-N/100} < Z^{-500}$ for suitable large N . In particular, $E_2 < E + Z^{-500} < E_* + Z^{-500} < 0 < E_0$. The spectrum of H_c below energy E_0 consists of the eigenvalues $E_0 - Z^2/4n^2$ for integers $n > l$, so we have $E_2 = E_0 - Z^2/4n^2$, $n > l$. Hence $|E - (E_0 - Z^2/4n^2)| < Z^{-500}$. Since $E < E_*$ we have $E_0 - Z^2/4n^2 < E_* + Z^{-500}$, so that by (c) we have also $E_0 - Z^2/4n^2 < E_*$, i.e., $n < Z/2(E_0 - E_*)^{1/2}$. Thus, $l < n < Z/2(E_0 - E_*)^{1/2}$ and $|E - (E_0 - Z^2/4n^2)| < Z^{-500}$. This completes the proof of (11).

Next we prove (12). Let u, \tilde{u} be the normalized eigenfunctions associated to E, \tilde{E} . Again we take $H_1 = H$, $E_1 = E$, $H_2 = H_c$ and note that the hypotheses of Lemma 2 hold, by virtue of (10). Hence the corollary to Lemma 2 produces a function w on $(0, \infty)$ with

$$\|u - w\| < Z^{-N/100}, \quad \|(H_c - E)w\| < Z^2 \cdot Z^{-N/100}. \quad (14)$$

Similarly, using \tilde{E} in place of E in the proof of (14), we get a function \tilde{w} on $(0, \infty)$, satisfying

$$\|\tilde{u} - \tilde{w}\| < Z^{-N/100}, \quad \|(H_c - E)\tilde{w}\| < Z^2 \cdot Z^{-N/100}. \quad (15)$$

Taking N suitably large and recalling that $\|u\| = \|\tilde{u}\| = 1$, $|E - (E_0 - Z^2/4n^2)|$, $|\tilde{E} - (E_0 - Z^2/4n^2)| < Z^{-500}$, we conclude that

$$\left\| \left(H_c - \left[E_0 - \frac{Z^2}{4n^2} \right] \right) w \right\|, \left\| \left(H_c - \left[E_0 - \frac{Z^2}{4n^2} \right] \right) \tilde{w} \right\| < 2 \cdot Z^{-500}. \quad (16)$$

Let ψ_n be the (normalized) eigenfunction of H_c corresponding to the eigenvalue $E_0 - Z^2/4n^2$. Then we can write

$$w = a\psi_n + \varphi \quad \text{with } a \in \mathbb{C}, \varphi \perp \psi_n; \quad (17)$$

and

$$\tilde{w} = \tilde{a}\psi + \tilde{\varphi} \quad \text{with } \tilde{a} \in \mathbb{C}, \tilde{\varphi} \perp \psi_n. \quad (18)$$

Note that the eigenvalue $E_0 - Z^2/4n^2$ is separated from the rest of the spectrum of H_c by at least

$$\begin{aligned} \frac{cZ^2}{n^3} &> cZ^2 \cdot \left[\frac{Z}{2(E_0 - E_*)^{1/2}} \right]^{-3} = c'Z^{-1}(E_0 - E_*)^{3/2} \\ &\geq c''Z^{-1}(Z^{4/3})^{3/2} \\ &= c''Z > 1. \end{aligned}$$

It follows by spectral theory that

$$\left\| \left(H_c - \left[E_0 - \frac{Z^2}{4n^2} \right] \right) w \right\| = \left\| \left(H_c - \left[E_0 - \frac{Z^2}{4n^2} \right] \right) \varphi \right\| \geq \|\varphi\|.$$

Hence (16) implies $\|\varphi\| \leq 2Z^{-500}$, and similarly $\|\tilde{\varphi}\| \leq 2Z^{-500}$. Thus, (17) and (18) show that

$$\|w - a\psi_n\|, \|\tilde{w} - \tilde{a}\psi_n\| \leq 2Z^{-500}. \quad (19)$$

On the other hand, $\|\psi_n\| = 1$ and $\|w\|, \|\tilde{w}\| \sim 1$ by (14) and (15), since $\|u\| = \|\tilde{u}\| = 1$. Therefore, (19) implies that $|a|, |\tilde{a}| \sim 1$. Putting this back into (19), we conclude that $\|w - a'\tilde{w}\| < CZ^{-500}$ for a complex number a' with $|a'| \sim 1$. Hence by (14) and (15) we have also $\|u - a'\tilde{u}\| = C'Z^{-500} \ll 1$. If E, \tilde{E} were distinct, then $u \perp \tilde{u}$, contradicting $\|u - a'\tilde{u}\| \ll 1$. Hence $E = \tilde{E}$, which completes the proof of (12).

It remains to prove (13). We invoke Lemma 2 with $H_1 = H_c$, $E_1 = E_0 - Z^2/4n^2 < E_*$ (by our restriction on n), $H_2 = H$. Again the hypotheses of Lemma 2 hold, by virtue of (10).

The conclusion of Lemma 2 gives us an $E \in \text{spectrum}(H)$ with $|E - (E_0 - Z^2/4n^2)| < |E_*| (|E_*| x_*^2)^{-N} < Z^2 \cdot Z^{-N/100} < Z^{-500}$ for suitably large N . In particular, $E < E_0 - Z^2/4n^2 + Z^{-500} < E_*$ by virtue of (c). Since the spectrum of H below E_* contains only eigenvalues, we see that $E < E_*$ is an eigenvalue of H satisfying $|E - (E_0 - Z^2/4n^2)| < Z^{-500}$. This completes the proof of (13) and with it, that of the lemma. ■

THE SECOND WKB EIGENVALUE SUM THEOREM

Suppose we are given a smooth potential $V(x)$ defined on $(0, \infty)$, an interval $I \subset (0, \infty)$ containing $\{V(x) < 0\}$, and two positive functions $S(x)$, $B(x)$ defined on I . We will say that $V(x)$ has an *exact Coulomb singularity* with parameters (l, E_0, Z, x_*) if the following conditions are satisfied:

$$(CS1) \quad V(x) = V_c(x) \equiv l(l+1)/x^2 + E_0 - Z/x \text{ for } 0 < x < 10x_*$$

$$(CS2) \quad S(x) = Z/x \text{ and } B(x) = x \text{ for } x \in I, x < 10x_*$$

$$(CS3) \quad l \text{ is an integer, and } 0 \leq l \leq \frac{1}{16}(Zx_*)^{1/2}$$

$$(CS4) \quad cZ^{4/3} < E_0 < CZ^{4/3}$$

$$(CS5) \quad Z^{-98/100} < x_* < Z^{-1/3 - 1/100}$$

(CS6) Z is greater than a large, positive constant determined by c and C in (CS4) above.

Strictly speaking, these conditions pertain to $V(x)$, $S(x)$, $B(x)$ rather than just $V(x)$, but this should cause no confusion.

In this section, we study potentials $V(x)$ satisfying the hypotheses (Z0)–(Z9) of the WKB Density Theorem, and also having an exact Coulomb singularity. We set $H = -d^2/dx^2 + V(x)$ on $(0, \infty)$ with Dirichlet boundary conditions. Our goal is to make an accurate computation of the sum $\text{sneg}(H)$ of the negative eigenvalues of H . The First WKB Eigenvalue Sum Theorem gives a formula for $\text{sneg}(H)$ with an error dominated by $A^{4\epsilon-2} |\min_{x \in (0, \infty)} V(x)|$. However, for $l \ll Z^{1/3}$, a Coulomb singularity has $\min_{x \in (0, \infty)} V(x)$ very large, so that the error grows too big. We will overcome this difficulty by comparing H with $H_c = -d^2/dx^2 + V_c(x)$ and using the known eigenvalues of H_c . Thus, instead of computing $\text{sneg}(H)$ directly, we will compute $\text{sneg}(H) - \text{sneg}(H_c)$ and get an error small compared to that in $\text{sneg}(H)$. This trick has been used repeatedly in the literature and is obvious to anyone interested in Coulomb singularities.

We will pick a suitable cutoff energy $E_* < 0$, and define:

$$X = \text{sum of the eigenvalues of } H \text{ belonging to } [E_*, 0] \quad (1)$$

$$X_c = \text{sum of the eigenvalues of } H_c \text{ belonging to } [E_*, 0] \quad (2)$$

$$Y = \text{sum of the eigenvalues of } H \text{ belonging to } (-\infty, E_*) \quad (3)$$

$$Y_c = \text{sum of the eigenvalues of } H_c \text{ belonging to } (-\infty, E_*). \quad (4)$$

So

$$\text{sneg}(H) - \text{sneg}(H_c) = X - X_c + (Y - Y_c). \quad (5)$$

Using the results of the previous section, we will show that $(Y - Y_c)$ is negligibly small. We will then compute X and X_c using the Lemma on Truncated Eigenvalue Sums. Putting these results into (5), we will get an accurate approximation to $\text{sneg}(H) - \text{sneg}(H_c)$.

To apply the Lemma on Truncated Eigenvalue Sums, we must pick E_* so that the phase $\phi(E_*)$ is close to a multiple of π . We compute $\phi(E)$ in the following elementary lemma, whose proof we include for the reader's convenience.

LEMMA 1. *Suppose E, l, Z are positive numbers, with $Z^2 > 4l(l+1)E$ so that $l(l+1)/x^2 + E - Z/x$ is negative somewhere in $(0, \infty)$. Then*

$$\int_0^\infty \left(- \left(\frac{l(l+1)}{x^2} + E - \frac{Z}{x} \right) \right)_+^{1/2} dx = \pi \left(\frac{Z}{2E^{1/2}} - \sqrt{l(l+1)} \right).$$

Proof. For $0 < a < b$, we compute

$$I = \int_a^b \sqrt{-(t-a)(t-b)} \frac{dt}{t}.$$

Multiplying together suitable branches of $(z-a)^{1/2}$ and $(z-b)^{1/2}$, we get a single-valued branch of $F(z) = \sqrt{(z-a)(z-b)}$ defined for $z \in \mathbb{C} \setminus [a, b]$, with the following properties:

$F(z) \sim z$ for $|z|$ large (rather than $F(z) \sim -z$);

$F(t+i0) = i\sqrt{-(t-a)(t-b)}$ and $F(t-i0) = i\sqrt{-(t-a)(t-b)}$ for $t \in [a, b]$;

$F(0) = -\sqrt{ab}$.

Then define contours $\gamma, \tilde{\gamma}, \Gamma$ in \mathbb{C} as follows:

$\tilde{\gamma}$ is a small circle about 0;

Γ is a huge circle about 0;

γ is a thin rectangle around the interval $[a, b]$.

We suppose that all these contours have counterclockwise orientation. Cauchy's theorem gives

$$\oint_{\Gamma} F(z) \frac{dz}{z} - \oint_{\gamma} F(z) \frac{dz}{z} - \oint_{\tilde{\gamma}} F(z) \frac{dz}{z} = 0. \quad (6)$$

If Γ has radius R , then

$$\begin{aligned} \oint_{\Gamma} F(z) \frac{dz}{z} &= \oint_{\Gamma} z \sqrt{\left(1 - \frac{a}{z}\right)\left(1 - \frac{b}{z}\right)} \cdot \frac{dz}{z} \\ &= \oint_{\Gamma} \left\{ 1 - \left(\frac{a+b}{2}\right) z^{-1} + O(|z|^{-2}) \right\} dz \\ &= -\left(\frac{a+b}{2}\right) \cdot 2\pi i + O(R^{-1}). \end{aligned} \quad (7)$$

We will let $R \rightarrow \infty$ in (6). Also,

$$\oint_{\tilde{\gamma}} F(z) \frac{dz}{z} = 2\pi i F(0) = -2\pi i \sqrt{ab}. \quad (8)$$

To evaluate $\oint_{\gamma} F(z) \frac{dz}{z}$, we write $\gamma = \{(t+i\varepsilon) \mid t \in [a-\varepsilon, b+\varepsilon]\} \cup \{(t-i\varepsilon) \mid t \in [a-\varepsilon, b+\varepsilon]\} \cup \{a-\varepsilon+it \mid t \in [-\varepsilon, +\varepsilon]\} \cup \{b+\varepsilon+it \mid t \in [-\varepsilon, \varepsilon]\}$ and let $\varepsilon \rightarrow 0+$. In view of our formulas for $F(t+i0)$, $F(t-i0)$, we get

$$\oint_{\gamma} F(z) \frac{dz}{z} = -2i \int_a^b \sqrt{-(t-a)(t-b)} \frac{dt}{t} = -2iI. \quad (9)$$

Putting (7), (8), (9) into (6) we get $I = \pi \cdot ((a+b)/2 - \sqrt{ab})$. Equivalently, if $\{t^2 - Qt + P < 0\}$ is a non-empty subinterval of $(0, \infty)$, then

$$\int_0^\infty (-(t^2 - Qt + P))_+^{1/2} \frac{dt}{t} = \pi \left(\frac{Q}{2} - \sqrt{P} \right).$$

This shows that

$$\begin{aligned} \int_0^\infty \left(- \left(\frac{P}{t^2} - \frac{Q}{t} + A \right) \right)_+^{1/2} dt &= \int_0^\infty (-(At^2 - Qt + P))_+^{1/2} \frac{dt}{t} \\ &= \pi \left(\frac{Q}{2\sqrt{A}} - \sqrt{P} \right) \\ &\quad \text{when } Q, A, P, Q^2 - 4AP > 0. \end{aligned} \quad (10)$$

The conclusion of the lemma is immediate from (10). ■

In view of Lemma 1, we will have

$$\phi_c(E_*) = \int_0^\infty (E_* - V_c(x))_+^{1/2} dx = \pi \left(\frac{Z}{2\sqrt{E_0 - E_*}} - \sqrt{l(l+1)} \right) = \pi m_* \quad (11)$$

for an integer m_* , provided we take

$$E_* = E_0 - \frac{Z^2}{4[m_* + \sqrt{l(l+1)}]^2}, \quad (12)$$

and provided

$$Z^2 > 4l(l+1)(E_0 - E_*). \quad (13)$$

We will pick

$$m_* = \text{largest integer} < \frac{1}{16} (Zx_*)^{1/2}, \quad (14)$$

and check that (13) holds, since

$$\begin{aligned} 4l(l+1)(E_0 - E_*) &= 4l(l+1) \cdot \frac{Z^2}{4[m_* + \sqrt{l(l+1)}]^2} \\ &< 4l(l+1) \cdot \frac{Z^2}{4[\sqrt{l(l+1)}]^2} = Z^2. \end{aligned}$$

LEMMA 2. Suppose $V(x)$ is smooth on $(0, \infty)$ and satisfies the hypotheses (Z0)–(Z9) and (CS1)–(CS6). Assume also that $l > Z^{1/K} + 10^9$. Define E_* by (12), (14). Then all the hypotheses of the Lemma on Low Eigenvalues in Coulomb Singularities are satisfied.

Proof. The hypotheses (a)–(f) of that lemma are as follows. Hypothesis (a) says that $cZ^{4/3} < E_0 < CZ^{4/3}$, which is (CS4). Hypothesis (b) says that $-Z^2 < E_* < -Z^{4/3}$. We have from (12) that $E_* > -Z^2/4[m_* + \sqrt{l(l+1)}]^2 > -Z^2$, which is the desired lower bound for E_* . To prove the desired upper bound for E_* , it is enough to check that

$$\frac{Z^2}{4[m_* + \sqrt{l(l+1)}]^2} > cZ^{4/3 + 1/100}, \quad (14bis)$$

i.e., $[m_* + \sqrt{l(l+1)}]^2 < CZ^{2/3 - 1/100}$, i.e., $m_*, l < CZ^{1/3 - 1/200}$. In fact we have by (14) and (CS3) that $m_*, l \leq \frac{1}{16}(Zx_*)^{1/2} < \frac{1}{16}(Z^{2/3 - 1/100})^{1/2} = \frac{1}{16}Z^{1/3 - 1/200}$ by (CS5). This completes the proof of (b).

Hypothesis (c) says that $\min_{n \in \mathbb{Z}} |E_* - (E_0 - Z^2/4n^2)| > Z^{-500}$, i.e., $\min_{n \in \mathbb{Z}} |Z^2/4[m_* + \sqrt{l(l+1)}]^2 - Z^2/4n^2| > Z^{-500}$, i.e.,

$$\min_{n \in \mathbb{Z}} |[m_* + \sqrt{l(l+1)}]^{-2} - n^{-2}| > 4Z^{-502}. \quad (15)$$

We have $\sqrt{l(l+1)} = l\sqrt{1+l^{-1}} = l + \frac{1}{2} + O(l^{-1})$. Since $l > Z^{1/K} + 10^9$, it follows that $|\sqrt{l(l+1)} - (l + \frac{1}{2})| < \frac{1}{100}$. Thus, $m_* + \sqrt{l(l+1)} = (m_* + l + \frac{1}{2}) + \xi$ with $|\xi| < \frac{1}{100}$. So in (15), the minimum is attained by $n = m_* + l$ or by $n = m_* + l + 1$. In either case we have $\frac{1}{4} < |n - (m_* + \sqrt{l(l+1)})| < 1$, so $|[m_* + \sqrt{l(l+1)}]^{-2} - n^{-2}| \sim n^{-3} \sim (m_* + l)^{-3} > c(Zx_*)^{-3/2}$ since we saw in the proof of (b) that $m_*, l \leq \frac{1}{16}(Zx_*)^{1/2}$. Thus,

$$\begin{aligned} \min_{n \in \mathbb{Z}} |[m_* + \sqrt{l(l+1)}]^{-2} - n^{-2}| &> c(Zx_*)^{-3/2} \\ &> c(Z^{2/3 - 1/100})^{-3/2} \gg 4Z^{-502}, \end{aligned}$$

completing the proof of (c).

Hypothesis (d) says that l is an integer and $0 \leq l \leq CZ^{1/3}$, which is immediate from (CS3) and (CS5).

Hypothesis (e) says that $x_* > Z^{-1}$ and $|E_*| x_*^2 > Z^{1/100}$. Now $x_* > Z^{-1}$ is contained in (CS5). From (CS4), (12), (14bis), we have $|E_*| \sim Z^2/4[m_* + l]^2$. Hence, since $l \leq \frac{1}{16}(Zx_*)^{1/2} \sim m_*$, we get $|E_*| x_*^2 \sim Z^2 x_*^2/[m_* + l]^2 \sim Z^2 x_*^2/m_*^2 \sim Z^2 x_*^2/(Zx_*)$ (by (14)) $= Zx_*$, so $|E_*| x_*^2 > cZ^{2/100}$ by (CS5). This completes the proof of (e).

Hypothesis (f) asserts that Z is large, which is just (CS6). Thus we have proven (a)–(f).

The remaining hypotheses of the Lemma on Low Eigenvalues in Coulomb Singularities are $V \in C^\infty(0, \infty)$, $V(x) = V_c(x)$ for $x \leq 10x_*$, and $V(x), V_c(x) > \frac{1}{2}E_*$ for $x > x_*$. Since (CS1)–(CS6) include the second of these assertions, and we are assuming the first, it remains only to show that

$$V(x), V_c(x) > \frac{1}{2}E_* \quad \text{for } x \geq x_*. \quad (16)$$

Now $V_c(x)$ is decreasing for $x < x_0 = 2l(l+1)/Z$ and increasing for $x > x_0$. We have $x_0 < 4l^2/Z \leq (4/Z) [\frac{1}{16}(Zx_*)^{1/2}]^2 = \frac{1}{64}x_*$, so $V_c(x)$ is increasing in $[x_*, \infty)$. Hence to check $V_c(x) > \frac{1}{2}E_*$ for $x \geq x_*$ it is enough to check that $V_c(x_*) = l(l+1)/x_*^2 + E_0 - Z/x_* > \frac{1}{2}E_*$. This follows from $\frac{1}{2}E_0 - Z/x_* > \frac{1}{2}E_*$, i.e., $Z/x_* < \frac{1}{2}(E_0 - E_*) = Z^2/8[m_* + \sqrt{l(l+1)}]^2$, i.e.,

$$[m_* + \sqrt{l(l+1)}]^2 < \left(\frac{Zx_*}{8}\right). \quad (17)$$

Since $m_* < \frac{1}{16}(Zx_*)^{1/2}$ and $\sqrt{l(l+1)} < l+1 \leq \frac{2}{16}(Zx_*)^{1/2}$, we have $[m_* + \sqrt{l(l+1)}]^2 < (\frac{3}{16})^2 Zx_* < Zx_*/8$, proving (17). So $V_c(x) > \frac{1}{2}E_*$ for $x \geq x_*$.

To handle $V(x)$, we argue as follows. There is a local minimum of $V_c(x)$ at $x = x_0 < x_*$; and $V_c(x_0) < 0$. Since $V(x) = V_c(x)$ for $x < x_*$, it follows that $V(x)$ has a local minimum at the same x_0 , and $V(x_0) < 0$. Hypotheses (Z0)–(Z9) show that x_0 is the global minimum of $V(x)$, which is also called x_0 in (Z0)–(Z9). From (Z0)–(Z9), we have $V(x)$ increasing in $[x_0, x_{rt}]$, and $V(x) > 0$ for $x > x_{rt}$. We know that $V(x_*) = V_c(x_*) > \frac{1}{2}E_*$. Now suppose $x > x_*$. Either $x \leq x_{rt}$, in which case $V(x) \geq V(x_*) > \frac{1}{2}E_*$; or else $x > x_{rt}$, in which case $V(x) > 0 > \frac{1}{2}E_*$. Thus $V(x) > \frac{1}{2}E_*$ for all $x \geq x_*$. The proof of Lemma 2 is complete. ■

By Lemma 2 and the Lemma on Low Eigenvalues in Coulomb Singularities, we have

$$|Y - Y_c| < Z^{-100}. \quad (\text{See (3), (4).}) \quad (18)$$

Next we prepare to compute X, X_c using the Lemma on Truncated Eigenvalue Sums.

LEMMA 3. *The hypotheses (Z0)–(Z9) of the WKB Density Theorem are satisfied when we take $V(x) = V_c(x)$; $S_c(x) = Z/x$, $B_c(x) = x$, $I_{\text{BVP}} = (0, \infty)$, $I_c = [10^{-9}(l^2/Z), 10^{+9}(Z/E_0)]$, with l and E_0 as in (CS1)–(CS6) and $l > Z^{(10^{-9})}$ and with $K = 10^9$ and \hat{c} suitably picked. Moreover, with $\lambda_c(x) = S_c^{1/2}(x) B_c(x)$ and $\Lambda_c = (\int_{V_c < 0} (dx/\lambda_c(x) B_c(x)))^{-1}$, we have $\Lambda_c \sim l$.*

Proof. We provide the elementary details for the reader's convenience. Hypothesis (Z0) says that $c < y/x < C$ and $c < (Z/x)/(Z/y) < C$ for

$|x - y| < c|x|$, which is obvious. Hypothesis (Z1) says that $|(d/dx)^\alpha \{l(l+1)/x^2 + E_0 - Z/x\}| \leq C_\alpha Z x^{-1-\alpha}$ in I_c . In fact,

$$\begin{aligned} \left| \left(\frac{d}{dx} \right)^\alpha \left\{ \frac{l(l+1)}{x^2} \right\} \right| &= c_\alpha l(l+1) x^{-2-\alpha} \\ &= c_\alpha \left\{ \frac{l(l+1)}{Z} x^{-1} \right\} Z x^{-1-\alpha} \\ &\leq C_\alpha Z x^{-1-\alpha} \quad \text{in } I_c. \end{aligned} \quad (19)$$

Also, when $\alpha=0$ we have $|(d/dx)^\alpha E_0| = E_0 \leq 10^{+9}(Z/x)$ in I_c , while $(d/dx)^\alpha E_0 = 0$ for $\alpha \neq 0$. So

$$\left| \left(\frac{d}{dx} \right)^\alpha E_0 \right| \leq 10^9 Z x^{-1-\alpha} \quad \text{in } I_c. \quad (20)$$

Evidently,

$$\left| \left(\frac{d}{dx} \right)^\alpha \left\{ \frac{Z}{x} \right\} \right| \leq C_\alpha Z x^{-1-\alpha}. \quad (21)$$

Estimates (19), (20), (21) imply (Z1).

Hypotheses (Z2) and (Z5) follow when we show that $\{x \in (0, \infty) \mid V_c(x) < 0\} = (x_{\text{left}}, x_{\text{rt}}) \subset I_c$ with $x_{\text{left}}, x_{\text{rt}} \in [10^{-6}(l^2/Z), 10^{+6}(Z/E_0)]$. In fact, $\{x^2 V_c(x) < 0\} = \{l(l+1) - Zx + E_0 x^2 < 0\}$ has the form $(x_{\text{left}}, x_{\text{rt}})$ with $-\infty < x_{\text{left}} < x_{\text{rt}} < +\infty$, since the discriminant $Z^2 - 4E_0 l(l+1)$ is positive. (Recall $E_0 < CZ^{4/3}$, $l \leq \frac{1}{16}(Zx_*)^{1/2} < CZ^{1/3-1/200}$, so

$$E_0 l(l+1) < CZ^{2-1/100}.) \quad (22)$$

We have $x_{\text{left}} = (Z - \sqrt{Z^2 - 4E_0 l(l+1)})/2E_0 > 0$, so $(x_{\text{left}}, x_{\text{rt}}) \subset (0, \infty)$. Note that

$$x_{\text{rt}} = \frac{Z + \sqrt{Z^2 - 4E_0 l(l+1)}}{2E_0} > \frac{Z}{2E_0} > cZ^{-1/3}. \quad (22\text{bis})$$

For $x \leq 10^{-6}(l^2/Z)$ we have $l(l+1)/x^2 + E_0 - Z/x > (Z/x^2)\{l(l+1)/Z - x\} > 0$, so $(x_{\text{left}}, x_{\text{rt}}) = \{l(l+1)/x^2 + E_0 - Z/x < 0\} \subset (10^{-6}(l^2/Z), \infty)$. For $x \geq 10^{+6}(Z/E_0)$ we have $l(l+1)/x^2 + E_0 - Z/x \geq (E_0/x)\{x - Z/E_0\} > 0$, so $(x_{\text{left}}, x_{\text{rt}}) \subset (10^{-6}(l^2/Z), 10^{+6}(Z/E_0))$, as claimed. We have proven (Z2) and (Z5).

Hypothesis (Z3) says that $V_c(x_0) < -c(Z/x_0)$, $V'(x_0) = 0$, $V''(x) > c(Z/x^3)$ for $|x - x_0| < c_1 x_0$. To find x_0 , set $V'_c(x) = -2l(l+1)/x^3 + Z/x^2$ equal to zero. Thus, $x_0 = 2l(l+1)/Z$, so $V_c(x_0) = l(l+1)/x_0^2 + E_0 - Z/x_0 = (Z/x_0)\{l(l+1)/Zx_0 + E_0 x_0/Z - 1\}$, i.e., $V_c(x_0) = (Z/x_0)\{\frac{1}{2} + E_0 x_0/Z - 1\} \sim -\frac{1}{2}(Z/x_0)$ since $E_0 x_0/Z \sim E_0 l(l+1)/Z^2 < CZ^{-1/100}$ by (22).

Also, for $|x - x_0| < \frac{1}{100}x_0$, we have

$$\begin{aligned} V_c''(x) &= \frac{6l(l+1)}{x^4} - \frac{2Z}{x^3} = \frac{6Z}{x^3} \left\{ \frac{l(l+1)}{Zx} - \frac{1}{3} \right\} > \frac{6Z}{x^3} \left\{ \frac{100}{101} \frac{l(l+1)}{Zx_0} - \frac{1}{3} \right\} \\ &= \frac{6Z}{x^3} \left\{ \frac{100}{101} \cdot \frac{1}{2} - \frac{1}{3} \right\} > \frac{cZ}{x^3}. \end{aligned}$$

This proves (Z3) with $c_1 = \frac{1}{100}$.

Hypothesis (Z4) says that

$$V_c'(x) < -\frac{cZ}{x^2} \quad \text{for } x_{\text{left}} < x < \frac{99}{100}x_0,$$

and

$$V_c'(x) > +\frac{cZ}{x^2} \quad \text{for } \frac{101}{100}x_0 < x < x_{\text{rt}}.$$

For $x < \frac{99}{100}x_0$ we have

$$V_c'(x) = \frac{Z}{x^2} \left\{ -\frac{2l(l+1)}{Zx} + 1 \right\} < \frac{Z}{x^2} \left\{ -\frac{100}{99} \cdot \frac{2l(l+1)}{Zx_0} + 1 \right\} = -\frac{1}{99} \frac{Z}{x^2}.$$

For $x > \frac{101}{100}x_0$ we have

$$\begin{aligned} V_c'(x) &= \frac{Z}{x^2} \left\{ -\frac{2l(l+1)}{Zx} + 1 \right\} > \frac{Z}{x^2} \left\{ -\frac{100}{101} \cdot \frac{2l(l+1)}{Zx_0} + 1 \right\} \\ &= +\frac{1}{101} \frac{Z}{x^2}, \end{aligned}$$

which proves (Z4).

Next we compute $\lambda_c(x)$ and $A_c = \left(\int_{x_{\text{left}}}^{x_{\text{rt}}} (dx/\lambda_c(x) B_c(x)) \right)^{-1}$. We have $\lambda_c(x) = (Z/x)^{1/2} \cdot x = Z^{1/2}x^{1/2}$, so

$$A_c^{-1} = \int_{x_{\text{left}}}^{x_{\text{rt}}} \frac{dx}{Z^{1/2}x^{3/2}} = (\text{const}) Z^{-1/2} \cdot (x_{\text{left}}^{-1/2} - x_{\text{rt}}^{-1/2}).$$

Now $x_{\text{left}} = \inf\{x \mid V_c(x) < 0\} \leq x_0$ since $V_c(x_0) < 0$ by (Z3). Thus $x_{\text{left}} < 2l(l+1)/Z$, while $x_{\text{rt}} > cZ^{-1/3}$. Since $l \leq \frac{1}{16}(Zx_*)^{1/2} < CZ^{1/3-1/200}$, we have $x_{\text{left}} \leq x_{\text{rt}}$ and therefore $A_c^{-1} \sim Z^{-1/2}x_{\text{left}}^{-1/2}$. We know that $10^{-6}l^2/Z < x_{\text{left}} < 2l(l+1)/Z$, so $x_{\text{left}} \sim l^2/Z$, and thus

$$A_c \sim Z^{1/2}x_{\text{left}}^{1/2} \sim l. \quad (23)$$

We have already checked (Z5).

Hypothesis (Z6) says that $V_c(x) \geq 1000/|x - x_{\text{left}}|^2$ for $0 < x < x_{\text{left}} - A_c^{(10^9)} x_{\text{left}}$, and that $V_c(x) \geq 1000/|x - x_{\text{rt}}|^2$ for $x_{\text{rt}} + A_c^{(10^9)} x_{\text{rt}} < x < \infty$.

The first assertion holds vacuously since $x_{\text{left}} - A_c^{(10^9)} x_{\text{left}} < 0$. The second assertion follows if we can prove that

$$V_c(x) > \frac{5000}{x^2} \quad \text{for } A_c^{(10^9)} x_{\text{rt}} < x < \infty. \quad (23\text{bis})$$

In fact, $V_c(x) = l(l+1)/x^2 + E_0 - Z/x \geq l(l+1)/x^2$ for $x > Z/E_0$, hence for $x > CZ^{-1/3}$, hence for $x > C'x_{\text{rt}}$, by virtue of (22bis). This implies (23bis) since we are taking here $l \geq Z^{(10^{-9})}$. So we have proven (Z6).

Hypothesis (Z7) says that $(\max I_c)/(\min I_c) < A_c^{10^9}$, i.e., $10^{18}(Z^2/l^2 E_0) < Cl^{(10^9)}$, by (23) and the definition of I_c . In fact, $10^{18}Z^2/l^2 E_0 < Z^2/E_0 \sim Z^{2/3}$ while $l^{(10^9)} > Z$. This proves (Z7).

Hypothesis (Z8) says that the constant \hat{c} , which has not yet been specified, is small enough. Just pick \hat{c} small, and (Z8) is satisfied.

Hypothesis (Z9) says that A_c is large, i.e., that l is large. Here we take $l > Z^{(10^{-9})}$ with Z very large. Hence l is large. So (Z9) is proven.

Thus (Z0)–(Z9) are all satisfied. ■

Now we know that both $V(x)$ and $V_c(x)$ satisfy (Z0)–(Z9). To apply the Lemma on Truncated Eigenvalue Sums, we still must check the assumptions made there on E_* , namely $(1/\pi)\phi(E_*)$, $(1/\pi)\phi_c(E_*)$ are integers, $\phi(E_*) < \phi(0) - 1$, $\phi_c(E_*) < \phi_c(0) - 1$. Here $\phi(E) = \int_0^\infty (E - V(x))_+^{1/2} dx$, $\phi_c(E) = \int_0^\infty (E - V_c(x))_+^{1/2} dx$. Equation (11) shows that $(1/\pi)\phi_c(E_*)$ is an integer. Moreover, for $E \leq \frac{1}{2}E_*$ we have $\phi_c(E) = \phi(E)$. In fact, Lemma 2 shows that $V(x)$, $V_c(x) > \frac{1}{2}E_*$ for $x \geq x_*$, hence the region of integration in the definitions of $\phi(E)$, $\phi_c(E)$ may be changed to $(0, x_*)$. In that region, $V(x) = V_c(x)$, so $\phi(E) = \phi_c(E)$ for $E \leq \frac{1}{2}E_*$. In particular, $(1/\pi)\phi(E)$ is an integer. To check that $\phi(E_*) < \phi(0) - 1$ and $\phi_c(E_*) < \phi_c(0) - 1$, we prove the stronger statements $\phi(E_*) < \phi(\frac{1}{2}E_*) - 1$ and $\phi_c(E_*) < \phi_c(\frac{1}{2}E_*) - 1$. These two statements are the same, so we may just look at ϕ_c , which is given by $(1/\pi)\phi_c(E) = Z/2(E_0 - E)^{1/2} - \sqrt{l(l+1)}$, by Lemma 1. Thus we must show that

$$\frac{\pi Z}{2}(E_0 - E_*)^{-1/2} < \frac{\pi Z}{2}\left(E_0 - \frac{1}{2}E_*\right)^{-1/2} - 1 \quad (24)$$

in order to verify the hypotheses of the Lemma on Truncated Eigenvalue Sums. From (14bis) we have $-E_* > cZ^{4/3 + 1/100}$, hence $-E_* \gg E_0$.

Therefore, $(E_0 - E_*) > \frac{3}{2}(E_0 - \frac{1}{2}E_*)$, so $(E_0 - \frac{1}{2}E_*)^{-1/2} - (E_0 - E_*)^{-1/2} > (E_0 - \frac{1}{2}E_*)^{-1/2} \cdot (1 - (\frac{3}{2})^{-1/2}) > c|E_*|^{-1/2}$. Hence, (24) holds provided $|E_*| < Z^{2-1/100}$. Since $m_* \sim (Zx_*)^{1/2}$, we have

$$\begin{aligned} -E_* &= -E_0 + \frac{Z^2}{4[m_* + \sqrt{l(l+1)}]^2} < \frac{Z^2}{4[m_* + \sqrt{l(l+1)}]^2} < \frac{Z^2}{4m_*^2} \\ &\sim \frac{Z}{x_*}. \end{aligned} \quad (25)$$

In particular, $|E_*| = -E_* < Z^{2-1/100}$ since $x_* > Z^{-1+1/100}$.

This completes the proof of (24) and allows us to use the Lemma on Truncated Eigenvalue Sums. We get

$$\begin{aligned} X &= -\frac{2}{3\pi} \int_0^\infty (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_0^\infty V''(x) \cdot (-V(x))_+^{-1/2} dx \\ &\quad + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \left\{ \frac{2}{3\pi} \int_0^\infty (E_* - V(x))_+^{3/2} dx \right. \\ &\quad - \frac{1}{24\pi} \int_0^\infty V''(x) (E_* - V(x))_+^{-1/2} dx + \frac{\pi}{24} (\phi'(E_*))^{-1} \\ &\quad \left. - \frac{E_*}{\pi} \phi(E_*) \right\} + \text{Error} \end{aligned} \quad (26)$$

with

$$|\text{Error}| \leq C_\# A^{4\epsilon-2} \inf_{\substack{x \in I \\ V(x) < E_*}} S(x). \quad (27)$$

$$\begin{aligned} X_c &= -\frac{2}{3\pi} \int_0^\infty (-V_c(x))_+^{3/2} dx + \frac{1}{24\pi} \int_0^\infty V_c''(x) \cdot (-V_c(x))_+^{-1/2} dx \\ &\quad + \frac{\pi}{2} (\phi'_c(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi_c(0) - \frac{1}{2} \right) \\ &\quad + \left\{ \frac{2}{3\pi} \int_0^\infty (E_* - V_c(x))_+^{3/2} dx - \frac{1}{24\pi} \int_0^\infty V_c''(x) \cdot (E_* - V_c(x))_+^{-1/2} dx \right. \\ &\quad \left. + \frac{\pi}{24} (\phi'_c(E_*))^{-1} - \frac{E_*}{\pi} \phi_c(E_*) \right\} + \text{Error}_c \end{aligned} \quad (28)$$

with

$$|\text{Error}_c| \leq C_\# A_c^{4\epsilon-2} \inf_{\substack{x \in I_c \\ V_c(x) < E_*}} S_c(x). \quad (29)$$

Here, $C_\#$ is as in the Lemma on Truncated Eigenvalue Sums.

We know that $\phi(E) = \phi_c(E)$ for $E < \frac{1}{2}E_*$, so $\phi(E_*) = \phi_c(E_*)$, $\phi'(E_*) = \phi'_c(E_*)$. Also, $V(x)$, $V_c(x) > E_*$ for $x \geq x_*$, and $V(x) = V_c(x)$ for $x < x_*$. Hence $(E_* - V(x))_+^{3/2} = (E_* - V_c(x))_+^{3/2}$ and $V''(x) \cdot (E_* - V(x))_+^{-1/2} = V_c''(x) \cdot (E_* - V(x))_+^{-1/2}$. This shows that the expressions in curly brackets in (26) and (28) are equal. So (26)–(29) imply

$$\begin{aligned} X - X_c &= -\frac{2}{3\pi} \int_0^\infty (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_0^\infty V''(x) \cdot (-V(x))_+^{-1/2} dx \\ &\quad + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) \\ &\quad + \frac{2}{3\pi} \int_0^\infty (-V_c(x))_+^{3/2} dx - \frac{1}{24\pi} \int_0^\infty V_c''(x) \cdot (-V_c(x))_+^{-1/2} dx \\ &\quad - \frac{\pi}{2} (\phi'_c(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi_c(0) - \frac{1}{2} \right) + \text{Error}_1, \end{aligned} \quad (30)$$

with

$$|\text{Error}_1| \leq C_\# A^{4\epsilon-2} \inf_{\substack{x \in I \\ V(x) < E_*}} S(x) + C_\# A_c^{4\epsilon-2} \inf_{\substack{x \in I_c \\ V_c(x) < E_*}} S_c(x). \quad (31)$$

Again recall that $V(x)$, $V_c(x) > \frac{1}{2}E_*$ for $x \geq x_*$. Hence the inf's in (31) occur over subsets of $(0, x_*)$, where $V(x) = V_c(x)$ and $S(x) = S_c(x)$. Also, from (Z0)–(Z9) applied to V and V_c , we know that $V(x) < E_*$ implies $x \in I$, and $V_c(x) < E_*$ implies $x \in I_c$. Therefore,

$$\inf_{\substack{x \in I \\ V(x) < E_*}} S(x) = \inf_{\substack{x \in I_c \\ V_c(x) < E_*}} S_c(x). \quad (32)$$

Let us compute the right side of (32). In fact, $\{V_c(x) < E_*\}$ is a non-empty interval (x_-, x_+) whose endpoints are obtained by solving $l(l+1)/x^2 + (E_0 - E_*) - Z/x = 0$. Hence

$$\begin{aligned} \inf_{\substack{x \in I_c \\ V_c(x) < E_*}} S_c(x) &= \inf_{(x_-, x_+)} \frac{Z}{x} = \frac{Z}{x_+} \\ &= Z \cdot \left[\frac{Z - \sqrt{Z^2 - 4l(l+1)(E_0 - E_*)}}{2l(l+1)} \right] \\ &= \frac{Z^2}{2l(l+1)} \left[1 - \left\{ 1 - \frac{4l(l+1)(E_0 - E_*)}{Z^2} \right\}^{1/2} \right] \\ &\leq \frac{Z^2}{2l(l+1)} \cdot \frac{Cl(l+1)(E_0 - E_*)}{Z^2} \\ &= C'(E_0 - E_*) \leq C'' \cdot (-E_*) \leq C''' \frac{Z}{x_*}, \end{aligned}$$

by (25). Putting this into (31), (32), we obtain

$$|\text{Error}_1| \leq C_{\#} (A^{4\epsilon-2} + l^{4\epsilon-2}) \cdot \frac{Z}{x_{\star}}, \quad (33)$$

since $A_c \sim l$. We can simplify (33) somewhat, because

$$A^{-1} \geq cl^{-1}. \quad (34)$$

To see this, write

$$\begin{aligned} A^{-1} &= \int_{\{V(x) < 0\}} \frac{dx}{\lambda(x) B(x)} \geq \int_{\{V(x) < 0\} \cap (0, x_{\star})} \frac{dx}{\lambda(x) B(x)} \\ &= \int_{\{V_c(x) < 0\} \cap (0, x_{\star})} \frac{dx}{\lambda_c(x) B_c(x)} \\ &\geq \int_{\{|x-x_0| < cx_0\} \cap (0, x_{\star})} \frac{dx}{Z^{1/2} x^{3/2}}. \end{aligned}$$

Here, $x_0 = 2l(l+1)/Z \leq 3l^2/Z \leq (3 \cdot [\frac{1}{16}(Zx_{\star})^{1/2}]^2)/Z = \frac{3}{256}x_{\star}$, so $\{|x-x_0| < cx_0\} \cap (0, x_{\star}) = \{|x-x_0| < cx_0\}$, and (35) becomes $A^{-1} \geq c(Zx_0)^{-1/2} = c(2l(l+1))^{-1/2} \geq c'l^{-1}$, proving (34). Hence, (33) simplifies to

$$|\text{Error}_1| \leq C_{\#} A^{4\epsilon-2} \frac{Z}{x_{\star}}. \quad (36)$$

Our basic results are (5), (18), (30), and (36). We rewrite (30) using the fact that $(1/\pi) \phi_c(E) = Z/2(E_0 - E)^{1/2} - \sqrt{l(l+1)}$. In particular,

$$(\pi/2)(\phi'_c(0))^{-1} = \frac{1}{2}((1/\pi) \phi'_c(0))^{-1} = \frac{1}{2}(Z/4(E_0)^{3/2})^{-1} = 2E_0^{3/2}/Z,$$

so

$$(\pi/2)(\phi'_c(0))^{-1} \tilde{\chi}((1/\pi) \phi_c(0) - \frac{1}{2}) = (2E_0^{3/2}/Z) \tilde{\chi}(Z/2E_0^{1/2} - \sqrt{l(l+1)} - \frac{1}{2}).$$

Also, note that $A^{4\epsilon-2}(Z/x_{\star}) \geq cl^{4\epsilon-2}(Z/x_{\star}) \geq Z^{-100}$.

Thus, (5), (18), (30), and (36) imply the following result, which strengthens the First WKB Eigenvalue Sum Theorem when $l \ll (Zx_{\star})^{1/2}$.

SECOND WKB EIGENVALUE SUM THEOREM. Suppose $V(x)$ is a smooth potential defined on $(0, \infty)$, $I \subset (0, \infty)$ is an interval containing $\{V(x) < 0\}$, and $S(x)$, $B(x)$ are positive functions defined on I . Also, let ϵ , K , N , l , E_0 , Z , x_{\star} be given.

Assume the hypotheses (Z0)–(Z9) of the WKB Density Theorem, and assume that $V(x)$ has an exact Coulomb singularity with parameters (l, E_0, Z, x_{\star}) . (That is, assume (CS1)–(CS6).) Finally, suppose $l > Z^{(10^{-9})}$.

Set $H = -d^2/dx^2 + V(x)$ on $(0, \infty)$ with Dirichlet boundary conditions. Then the sum of the negative eigenvalues of H is given by the equation

$$\begin{aligned} & \text{sneq}(H) - \text{sneq}(H_c) \\ &= -\frac{2}{3\pi} \int_0^\infty (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_0^\infty V''(x) \cdot (-V(x))_+^{-1/2} dx \\ &+ \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \frac{2}{3\pi} \int_0^\infty (-V_c(x))_+^{3/2} dx \\ &- \frac{1}{24\pi} \int_0^\infty V_c''(x) \cdot (-V_c(x))_+^{-1/2} dx - \frac{2E_0^{3/2}}{Z} \tilde{\chi} \left(\frac{Z}{2E_0^{1/2}} - \sqrt{l(l+1)} - \frac{1}{2} \right) \\ &+ \text{Error}, \end{aligned}$$

with $H_c = -d^2/dx^2 + V_c(x)$ on $(0, \infty)$ subject to Dirichlet boundary conditions, $V_c(x) = l(l+1)/x^2 + E_0 - Z/x$, and $|\text{Error}| \leq A^{5\epsilon-2}(Z/x_*)$. Recall that $\phi(E) = \int_0^\infty (E - V(x))_+^{1/2} dx$, $\phi'(E) = \frac{1}{2} \int_0^\infty (E - V(x))_+^{-1/2} dx$, $\tilde{\chi}(t) = \min_{k \in \mathbb{Z}} |t - k - \frac{1}{2}|^2 - \frac{1}{12}$.

EIGENVALUE SUMS FOR DEGENERATE POTENTIALS

In this section, we derive crude versions of the Second WKB Eigenvalue Sum Theorem that hold under weakened hypotheses. The idea is as follows. Suppose $H = -d^2/dx^2 + V(x)$ on $(0, \infty)$ (Dirichlet b.c.), with $V(x)$ having an exact Coulomb singularity (i.e., smooth on $(0, \infty)$ and satisfying (CS1)–(CS6) for suitable (l, E_0, Z, x_*)). We pick $E_* \in (-8Z/x_*, -8Z/3x_*)$ so that

$$\min_{n \in \mathbb{Z}} \left| E_* - \left(E_0 - \frac{Z^2}{4n^2} \right) \right| > Z^{-500}.$$

If $V(x) > \frac{1}{2}E_*$ for $x > x_*$, then the hypotheses of the Lemma on Low Eigenvalues in Coulomb Singularities apply. Hence, with $H_c = -d^2/dx^2 + V_c(x)$, $V_c(x) = l(l+1)/x^2 + E_0 - Z/x$, we have

$$\left| \sum_k (E_* - E_k)_+ - \sum_k (E_* - E_k^c)_+ \right| \leq Z^{-100}, \quad (1)$$

where $\{E_k\}$ are the negative eigenvalues of H and $\{E_k^c\}$ are those of H_c .

Now let

$\mathcal{N}(E)$ = number of eigenvalues of H which are $< E$

$\mathcal{N}_c(E)$ = number of eigenvalues of H_c which are $< E$,

and suppose we can prove that

$$\left| \mathcal{N}(E) - \frac{1}{\pi} \int_0^\infty (E - V(x))_+^{1/2} dx \right| \leq C \quad \text{for } E \in [E_*, 0]. \quad (2)$$

We have also

$$\left| \mathcal{N}_c(E) - \frac{1}{\pi} \int_0^\infty (E - V_c(x))_+^{1/2} dx \right| \leq C \quad \text{for } E \in [E_*, 0], \quad (3)$$

by virtue of our explicit knowledge of the eigenvalues of H_c and of the integral $(1/\pi) \int_0^\infty (E - V_c(x))_+^{1/2} dx = Z/2(E_0 - E)^{1/2} - \sqrt{l(l+1)}$.

For each $E_k < 0$, we have

$$E_k + (E_* - E_k)_+ = - \int_{E_*}^\infty \chi_{E_k < E} dE.$$

Summing over k , we get

$$\text{sneg}(H) + \sum_k (E_* - E_k)_+ = - \int_{E_*}^\infty \mathcal{N}(E) dE. \quad (4)$$

Similarly,

$$\text{sneg}(H_c) + \sum_k (E_* - E_k^c)_+ = - \int_{E_*}^0 \mathcal{N}_c(E) dE. \quad (5)$$

Subtracting (5) from (4) and using (1), (2), (3), we conclude that

$$\begin{aligned} \text{sneg}(H) - \text{sneg}(H_c) &= -\frac{1}{\pi} \int_{E_*}^0 \int_0^\infty (E - V(x))_+^{1/2} dx dE \\ &\quad + \frac{1}{\pi} \int_{E_*}^0 \int_0^\infty (E - V_c(x))_+^{1/2} dx dE + \text{Error}, \end{aligned} \quad (6)$$

$$|\text{Error}| < Z^{-100} + C |E_*| < C' \frac{Z}{x_*}. \quad (7)$$

Note that

$$\begin{aligned} \frac{1}{\pi} \int_{E_*}^0 \int_0^\infty (E - V(x))_+^{1/2} dx dE &= \frac{1}{\pi} \int_0^\infty \left\{ \int_{E_*}^0 (E - V(x))_+^{1/2} dE \right\} dx \\ &= \frac{2}{3\pi} \int_0^\infty \{ (-V(x))_+^{3/2} - (E_* - V(x))_+^{3/2} \} dx, \end{aligned}$$

and similarly for $V_c(x)$. Since $V(x) = V_c(x)$ for $x \leq x_*$, while $V(x)$, $V_c(x) > E_*$ for $x > x_*$, it follows that $(E_* - V(x))_+^{3/2} = (E_* - V_c(x))_+^{3/2}$ everywhere. Thus

$$\begin{aligned} & -\frac{1}{\pi} \int_{E_*}^0 \int_0^\infty (E - V(x))_+^{1/2} dx dE + \frac{1}{\pi} \int_{E_*}^0 \int_0^\infty (E - V_c(x))_+^{1/2} dx dE \\ & = -\frac{2}{3\pi} \int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\} dx. \end{aligned} \quad (8)$$

Note that the quantity in curly brackets vanishes near $x=0$, although $(-V(x))^{3/2}$ fails to be integrable when $V(x) \sim -Z/x$ near $x=0$.

Putting (8) into (6), (7), we get

$$\text{sneg}(H) - \text{sneg}(H_c) = -\frac{2}{3\pi} \int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\} dx + \text{Error} \quad (9)$$

with

$$|\text{Error}| < C \frac{Z}{x_*}. \quad (10)$$

So proving (9), (10) for potentials with exact Coulomb singularities is reduced to proving (2), which concerns the number of eigenvalues $< E$. To establish (2), we use our previous results on degenerate potentials.

THEOREM 1. Suppose $V(x)$, $S(x)$, $B(x)$ satisfy hypotheses $(Z\bar{0})$ – $(Z\bar{8})$, and also suppose $V(x)$ has an exact Coulomb singularity (CS1)–(CS6) for a given (l, E_0, Z, x_*) . Assume $E_{\text{crit}} < -3Z/x_*$, with E_{crit} as in $(Z\bar{0})$ – $(Z\bar{8})$. Assume also $V(x) > -\frac{4}{3}(Z/x_*)$ for $x > x_*$. Set $V_c(x) = l(l+1)/x^2 + E_0 - Z/x$. Then for $H = -d^2/dx^2 + V(x)$, $H_c = -d^2/dx^2 + V_c(x)$ on $(0, \infty)$ with Dirichlet boundary conditions, we have

$$\text{sneg}(H) - \text{sneg}(H_c) = -\frac{2}{3\pi} \int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\} dx + \text{Error},$$

with $|\text{Error}| < C(Z/x_*)$. Here, C depends only on the constants in $(Z\bar{0})$ – $(Z\bar{8})$ and in (CS1)–(CS6).

Proof. Take $E_* \in (-3Z/x_*, -\frac{8}{3}(Z/x_*))$, such that $\min_n |E_* - (E_0 - Z^2/4n^2)| > Z^{-500}$. Then $E_{\text{crit}} \leq E_* \leq 0$, and $V(x) > \frac{1}{2}E_*$ for $x > x_*$. For $E \in [E_{\text{crit}}, 0]$, and hence for $E \in [E_*, 0]$, Theorem D2 from the section on WKB theory with weak turning points implies (2). Hence (9), (10) follow, and Theorem 1 is proven. ■

For Hypotheses $(Z\bar{0})$ – $(Z\bar{8})$, see Review of Earlier Results, Section G.

THEOREM 2. Suppose $V(x)$, $S(x)$, $B(x)$ satisfy hypotheses $(Z\hat{0})$ – $(Z\hat{9})$, and suppose also that $V(x)$ has an exact Coulomb singularity $(CS1)$ – $(CS6)$ for a given (l, E_0, Z, x_*) . Assume $V(x) > -2Z/x_*$ for $x > x_*$. Assume $V(2x_0) < -8Z/x_*$, and assume

$$\int_0^\infty \{(-V(x))_+^{1/2} - (V(\frac{1}{2}x_1) - V(x))_+^{1/2}\} dx \leq C, \quad (11)$$

where x_0, x_1 are as in $(Z\hat{0})$ – $(Z\hat{9})$. Set $V_c(x) = l(l+1)/x^2 + E_0 - Z/x$. Then for $H = -d^2/dx^2 + V(x)$, $H_c = -d^2/dx^2 + V_c(x)$ on $(0, \infty)$ with Dirichlet boundary conditions, we have

$$\text{sneg}(H) - \text{sneg}(H_c) = -\frac{2}{3\pi} \int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\} dx + \text{Error},$$

with $|\text{Error}| < C(Z/x_*)$. Here C depends only on C in (11), and on the constants in $(Z\hat{0})$ – $(Z\hat{9})$ and in $(CS1)$ – $(CS6)$.

For hypotheses $(Z\hat{0})$ – $(Z\hat{9})$ see Review of Earlier Results, Section E.

Proof. Take $E_* \in (-8Z/x_*, -4Z/x_*)$, such that $\min_n |E_* - (E_0 - Z^2/4n^2)| > Z^{-500}$. Then $V(x) > \frac{1}{2}E_*$ for $x > x_*$, and $V(2x_0) < E_*$. It is enough to establish (2) for $E \in [E_*, 0]$. Hence, it is enough to establish (2) for $E \in [V(2x_0), 0]$. Since $\mathcal{N}(E)$ is monotone, (11) shows that it is enough to establish (2) for $E=0$ and for $E \in [V(2x_0), V(\frac{1}{2}x_1)]$. These follow from Theorem D1 in the section on WKB theory with weak turning points, together with Lemmas E1 and E2 from Review of Earlier Results, Section E. Note that the semiclassical approximation to $\mathcal{N}(E)$ is given in Theorem D1 as $(1/\pi) \int_{I_{\text{center}}} (E - V(x))_+^{1/2} dx$. Since $\int_{(0, \infty) \setminus I_{\text{center}}} (E - V(x))_+^{1/2} dx < C$ here, the conclusion of Theorem D1 is equivalent to (2). ■

THEOREM 3. Suppose $V(x)$, $S(x)$, $B(x)$ satisfy hypotheses $(Z0^+)$ – $(Z7^+)$. Suppose also that $V(x)$ has an exact Coulomb singularity $(CS1)$ – $(CS6)$ for given (l, E_0, Z, x_*) . [We take the same x_* in $(Z0^+)$ – $(Z7^+)$ as in $(CS1)$ – $(CS6)$.] We make the following additional assumptions:

$$V\left(\frac{x_*}{10}\right) < -\frac{3Z}{x_*} \quad (12)$$

$$V(x) > -\frac{4}{3} \frac{Z}{x_*} \quad \text{for } x > x_* \quad (13)$$

$$V(x) > -Cx_{\text{big}}^2 x^{-4} \quad \text{for } x > x_{\text{big}}. \quad (14)$$

$$\int_0^\infty \left\{ (-V(x))_+^{1/2} - \left(V\left(\frac{1}{4}x_1\right) - V(x) \right)_+^{1/2} \right\} dx < C, \quad (15)$$

with x_1 as in $(Z0^+)$ – $(Z7^+)$.

Set $V_c(x) = l(l+1)/x^2 + E_0 - Z/x$. Then for $H = -d^2/dx^2 + V(x)$, $H_c = -d^2/dx^2 + V_c(x)$ on $(0, \infty)$ with Dirichlet boundary conditions, we have

$$\text{sneg}(H) - \text{sneg}(H_c) = -\frac{2}{3\pi} \int_0^\infty \{(-V(x))_+^{3/2} - (-V_c(x))_+^{3/2}\} dx + \text{Error},$$

with $|\text{Error}| < CZ/x_*$. Here, C depends only on \underline{C} in (15), and on the constants in $(Z0^+)$ – $(Z7^+)$, $(CS1)$ – $(CS6)$, and (14).

For hypotheses $(Z0^+)$ – $(Z7^+)$, see Review of Earlier Results, Section F.

Proof. Take $E_* \in (-3Z/x_*, -\frac{8}{3}(Z/x_*))$, with $\min_n |E_* - (E_0 - Z^2/4n^2)| > Z^{-500}$. Then $V(x) > \frac{1}{2}E_*$ for $x > x_*$, by (13). As before, it is enough to prove (2), and by (15) we need only look at $E = 0$ and at $E \in [E_*, V(\frac{1}{4}x_1)]$. By (12), $[E_*, V(\frac{1}{2}x_1)] \subset [V(\frac{1}{10}x_*), V(\frac{1}{4}x_1)]$. For $E \in [V(\frac{1}{10}x_*), V(\frac{1}{4}x_1)]$, Lemma F2 in Review of Earlier Results applies. Therefore, Theorem D1 in the section on WKB theory with weak turning points gives

$$\mathcal{N}(E) = O(1) + \frac{1}{\pi} \int_{x_0}^{\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}} (E - V(x))_+^{1/2} dx, \quad (16)$$

where $V(\tilde{x}) = E$, $\frac{1}{10}x_* \leq \tilde{x} \leq \frac{1}{4}x_1$. From $(Z4^+)$ we have

$$\int_0^{x_0} (E - V(x))_+^{1/2} dx \leq \int_0^{x_0} (-V(x))_+^{1/2} dx = O(1). \quad (17)$$

Lemma F2 shows that $|E - V(x)| \leq C(\lambda^{-2/3}(\tilde{x}) \cdot \tilde{x})^{-2}$ for $x \in [\tilde{x} - \hat{C}_\# \lambda^{-2/3}(\tilde{x}) \cdot \tilde{x}, \tilde{x}]$, because that is contained in condition $(H\hat{5})$ in the conclusion of Lemma F2. This and (16), (17) imply

$$\mathcal{N}(E) = O(1) + \frac{1}{\pi} \int_0^{\tilde{x}} (E - V(x))_+^{1/2} dx. \quad (18)$$

Now $(Z2^+)$, $(Z5^+)$ show that $V(x)$ is increasing in $[\tilde{x}, x_{\text{big}}] \subset [x_0, x_1] \cup [x_1, x_{\text{big}}]$. Since $V(\tilde{x}) = E$, we conclude that $V(x) > E$ for $x \in [\tilde{x}, x_{\text{big}}]$. Hence, (18) is equivalent to

$$\mathcal{N}(E) = O(1) + \frac{1}{\pi} \int_0^{x_{\text{big}}} (E - V(x))_+^{1/2} dx. \quad (19)$$

From (14) we get $(E - V(x))_+ \leq (-V(x))_+ \leq Cx_{\text{big}}^2 x^{-4}$ for $x > x_{\text{big}}$, so $\int_{x_{\text{big}}}^\infty (E - V(x))_+^{1/2} dx = O(1)$, and (19) implies $\mathcal{N}(E) = O(1) + (1/\pi) \int_0^\infty (E - V(x))_+^{1/2} dx$, which is (2).

Thus we have proven (2) for $E \in [V(\frac{1}{10}x_*), V(\frac{1}{4}x_1)]$.

To complete the proof of Theorem 3, we must check (2) for $E=0$. Lemma F1 and Theorem D1 from the section on WKB theory with weak turning points together yield

$$\mathcal{N}(0) = O(1) + \frac{1}{\pi} \int_{x_0}^{x_1} (-V(x))_+^{1/2} dx. \quad (20)$$

From $(Z4^+)$ we get $\int_0^{x_0} (-V(x))_+^{1/2} dx = O(1)$.

From $(Z5^+)$ we get $\int_{x_1}^{x_{\text{big}}} (-V(x))_+^{1/2} dx = O(1)$.

From (14) we get $\int_{x_{\text{big}}}^{\infty} (-V(x))_+^{1/2} dx = O(1)$. Therefore, (20) implies $\mathcal{N}(0) = O(1) + (1/\pi) \int_0^{\infty} (-V(x))_+^{1/2} dx$, which is (2). The proof of Theorem 3 is complete. ■

THE THIRD WKB EIGENVALUE SUM THEOREM

Suppose we are in the setting of the WKB Theorem on Low Eigenvalues, with $E_{\infty} = 0$. We assume the hypotheses $(H0^*)$ – $(H6^*)$ of that theorem, and we make the further assumption that

$$-\lambda^{-3\epsilon} S < V(x_0) < 0. \quad (1)$$

Our goal is to control the sum of the negative eigenvalues of $H = -d^2/dx^2 + V(x)$ on I_{BVP} , with Dirichlet or Neumann boundary conditions.

In view of (1), the minimum of the potential is relatively small, so the hypotheses of the First WKB Eigenvalue Sum Theorem break down. However, the ideas used in its proof apply here, with fewer technical difficulties.

We begin by quoting Lemma B1 from the section the WKB Theorem on Low Eigenvalues. For $E \in (E_{\min}, E_{\max}] \equiv (V(x_0), V(x_0) + \lambda^{-2\epsilon} S]$, we have phases $\phi(E)$, $\psi(E)$ with

$$\left| \left(\frac{d}{dE} \right)^{\beta} \phi(E) \right| \leq C_{\#}^{\beta} \lambda S^{-\beta} \quad (2)$$

$$\left| \left(\frac{d}{dE} \right)^{\beta} \psi(E) \right| \leq C_{\#}^{\beta} \lambda^{-1} S^{-\beta} \quad (3)$$

$$\frac{d}{dE} \phi(E) \geq c_{\#} \lambda S^{-1}. \quad (4)$$

Throughout this section, $c_{\#}$, $C_{\#}$, etc., denote constants depending only on c , c' , c'' , C_{α} , ε , K , N in (H0*)–(H6*). Setting $\Phi(E) = (1/\pi)\phi(E) + (1/48\pi)\psi(E) - \frac{1}{2}$, we get

$$\left| \left(\frac{d}{dE} \right)^{\beta} \Phi(E) \right| \leq C_{\#}^{\beta} \lambda S^{-\beta} \quad (5)$$

$$\left(\frac{d}{dE} \right) \Phi(E) \geq c_{\#} \lambda S^{-1} \quad (6)$$

for $E \in (E_{\min}, E_{\max}]$.

Let $(t_{\min}, t_{\max}] = \Phi((E_{\min}, E_{\max}])$. Then for $t \in (t_{\min}, t_{\max}]$, the equation $\Phi(E) = t$ has a unique solution $E = E(t) \in (E_{\min}, E_{\max}]$, with

$$\left| \left(\frac{d}{dE} \right)^m E(t) \right| \leq C_{\#}^m S \lambda^{-m} \quad (7)$$

$$\frac{d}{dt} E(t) \geq c_{\#} S \lambda^{-1}. \quad (8)$$

We pick $E_{*} \in (V(x_0), 0)$ with $\phi(E_{*}) < \lambda^{-2}$, and with E_{*} less than all the eigenvalues of H .

Let $a_0 = \Phi(E_{*})$, and let $b_0 = \Phi(0)$. Since $E_{\max} = V(x_0) + \lambda^{-2\varepsilon} S > \frac{1}{2} \lambda^{-2\varepsilon} S$ by (1), it follows from (6) that

$$t_{\max} = \Phi(E_{\max}) \geq c_{\#} \lambda S^{-1} \cdot \frac{1}{2} \lambda^{-2\varepsilon} S + \Phi(0) = c'_{\#} \lambda^{1-2\varepsilon} + b_0.$$

Thus,

$$t_{\min} < a_0 < b_0 < t_{\max} - c_{\#} \lambda^{1-2\varepsilon}, \quad (9)$$

and

$$|a_0 + \frac{1}{2}| \leq C_{\#} \lambda^{-1}. \quad (10)$$

Next, we control the non-positive eigenvalues of H , by applying the WKB Theorem on Low Eigenvalues. The non-positive eigenvalues are $E_0, E_1, \dots, E_{k_{\max}}$, with

$$|\Phi(E_k) - k| < C_{\#} \lambda^{4\varepsilon-2}, \quad (11)$$

$$\{0, \dots, k_{\max}\} = \mathbb{Z} \cap [a, b], \quad (12)$$

$$a = a_0 \quad (13)$$

$$b = b_0 + w_{hi} \quad (14)$$

$$|w_{hi}| < C_{\#} \lambda^{4\varepsilon-2} \quad (15)$$

$$w_{hi} = 0 \quad \text{if} \quad \min_{k \in \mathbb{Z}} |b_0 - k| > C_{\#} \lambda^{4\varepsilon-2}. \quad (16)$$

We leave to the reader the task of deducing (12)–(16) from the WKB Theorem on Low Eigenvalues. Note that $a \leq b$. If $|a_0 - b_0| > C_{\#} \lambda^{4\epsilon-2}$, then this follows from (9), (13), (14), (15). If $|a_0 - b_0| \leq C_{\#} \lambda^{4\epsilon-2}$, then (10) gives $|b_0 + \frac{1}{2}| \leq C_{\#} \lambda^{-1} \ll 1$, so $\min_{k \in \mathbb{Z}} |b_0 - k| > \frac{1}{2} - C_{\#} \lambda^{-1} > \frac{1}{4}$, and (16) gives $w_{hi} = 0$. Then (9), (13), (14) show that $a \leq b$. Hence $a \leq b$ in all cases.

We use (11)–(15) and the Lemma on Riemann Sums to control

$$\text{sneg}(H) = \sum_{k=0}^{k_{\max}} E_k.$$

Since $t_{\min} < a_0 = a \leq b \leq b_0 + 1 \leq t_{\max}$ by (9), we have $[a, b] \subset (t_{\min}, t_{\max}]$, so that each k from 0 to k_{\max} belongs to the domain of $E(t)$ by (12). Hence for $0 \leq k \leq k_{\max}$, we have

$$\begin{aligned} |E_k - E(k)| &\leq \max_{t \in (t_{\min}, t_{\max}]} \left| \frac{d}{dt} E(t) \right| \cdot |\Phi(E_k) - \Phi(E(k))| \\ &= \max_{t \in (t_{\min}, t_{\max}]} \left| \frac{d}{dt} E(t) \right| \cdot |\Phi(E_k) - k| \leq C_{\#} S \lambda^{-1} \cdot C_{\#} \lambda^{4\epsilon-2} \end{aligned}$$

by (7), (11). Thus,

$$|E_k - E(k)| \leq C_{\#} S \lambda^{4\epsilon-3} \quad \text{for } 0 \leq k \leq k_{\max}. \quad (17)$$

We estimate the number of E_k . Evidently,

$$\begin{aligned} k_{\max} &\leq C_{\#} + (b - a) \leq C'_{\#} + (b_0 - a_0) \\ &\leq C'_{\#} + C_{\#} \max_{E \in (E_{\min}, E_{\max}]} \left| \frac{d\Phi}{dE}(E) \right| \cdot |E_{\max} - E_{\min}| \\ &\leq C'_{\#} + C_{\#} \lambda S^{-1} \cdot \lambda^{-2\epsilon} S \leq C''_{\#} \lambda^{1-2\epsilon}. \end{aligned}$$

Thus,

$$0 \leq b - a \leq C_{\#} \lambda^{1-2\epsilon}, \quad (18)$$

and

$$k_{\max} \leq C_{\#} \lambda^{1-2\epsilon}. \quad (19)$$

From (12), (17), (19), we see that $|\sum_{k=0}^{k_{\max}} E_k - \sum_{k \in \mathbb{Z} \cap [a, b]} E(k)| \leq C_{\#} S \lambda^{2\epsilon-2}$, i.e.,

$$\left| \text{sneg}(H) - \sum_{k \in [a, b] \cap \mathbb{Z}} E(k) \right| \leq C_{\#} S \lambda^{2\epsilon-2}. \quad (20)$$

Since $a \leq b$, estimates (7) and the Lemma on Riemann Sums yield

$$\begin{aligned} \sum_{k \in [a, b] \cap \mathbb{Z}} E(k) &= \int_a^b E(t) dt - E(b) \chi_-(b) - E(a) \chi_+(a) \\ &\quad + \frac{1}{2} \frac{dE(t)}{dt} \Big|_{t=b} \tilde{\chi}(b) - \frac{1}{2} \frac{dE(t)}{dt} \Big|_{t=a} \tilde{\chi}(a) + \text{Error}_0. \end{aligned} \quad (21)$$

with

$$|\text{Error}_0| \leq C_{\#} S \lambda^{-2} + C_{\#} \int_a^b S \lambda^{-N} dt \leq C'_{\#} S \lambda^{-2}, \quad (22)$$

by (18). Combining (20)–(22), we get

$$\begin{aligned} \text{neg}(H) &= \int_a^b E(t) dt - E(b) \chi_-(b) - E(a) \chi_+(a) \\ &\quad + \frac{1}{2} E'(b) \tilde{\chi}(b) - \frac{1}{2} E'(a) \tilde{\chi}(a) + \text{Error}_1 \end{aligned} \quad (23)$$

with

$$|\text{Error}_1| \leq C_{\#} S \lambda^{2\varepsilon-2}. \quad (24)$$

We study in turn each of the terms on the right in (23). Regarding the integral, we have

$$\begin{aligned} \left| \int_a^b E(t) dt - \int_{a_0}^{b_0} E(t) dt \right| &\leq \max_{t \in (t_{\min}, t_{\max}]} |E(t)| \cdot |b - b_0| \\ &\leq c_{\#} \lambda^{-2\varepsilon} S \cdot C_{\#} \lambda^{4\varepsilon-2} = C_{\#} \lambda^{2\varepsilon-2} S, \end{aligned}$$

i.e.,

$$\int_a^b E(t) dt = \int_{a_0}^{b_0} E(t) dt + \text{Error}_2, \quad \text{with } |\text{Error}_2| \leq C_{\#} S \lambda^{2\varepsilon-2}. \quad (25)$$

We postpone discussion of the integral from a_0 to b_0 .

Next, $|E(b)| = |E(b) - E(b_0)| \leq \max_{t \in (t_{\min}, t_{\max}]} |dE(t)/dt| \cdot |b - b_0| \leq C_{\#} S \lambda^{-1} \cdot C_{\#} \lambda^{4\varepsilon-2} \leq C_{\#} S \lambda^{4\varepsilon-3}$. Throwing away information, we conclude that

$$|E(b) \chi_-(b)| \leq C_{\#} S \lambda^{2\varepsilon-2}. \quad (26)$$

Next we study $-E(a) \chi_+(a)$. Recall that $E(a) = E_{\star}$ and $a = \Phi(E_{\star}) = (1/\pi) \phi(E_{\star}) + (1/48\pi) \psi(E_{\star}) - \frac{1}{2} = \xi + (1/48\pi) \psi(E_{\star}) - \frac{1}{2}$ with $|\xi| \leq \lambda^{-2}$,

$|\psi(E_*)| \leq C_{\#} \lambda^{-1}$. Hence, $k \equiv (\text{smallest integer } \geq a) = 0$, so $\chi_+(a) = k - a - \frac{1}{2} = -\xi - (1/48\pi) \psi(E_*)$ with $|\xi| \leq \lambda^{-2}$. We conclude that

$$-E(a) \chi_+(a) = + \frac{E_* \psi(E_*)}{48\pi} + \text{Error}_3, \quad (27)$$

with

$$|\text{Error}_3| \leq \lambda^{-2} |E_*| \leq C_{\#} S \lambda^{-2} \ll S \lambda^{2\epsilon-2}. \quad (28)$$

Next we study $E'(b) \tilde{\chi}(b)$. Recall that $\tilde{\chi}(\cdot)$ is Lipschitz continuous, and that $b = (1/\pi) \phi(0) + (1/48\pi) \psi(0) - \frac{1}{2} + w_{hi} = (1/\pi) \phi(0) - \frac{1}{2} + \xi$ with $|\xi| \leq C_{\#} \lambda^{-1}$. Hence

$$\left| \tilde{\chi}(b) - \tilde{\chi}\left(\frac{1}{\pi} \phi(0) - \frac{1}{2}\right) \right| \leq C_{\#} \lambda^{-1}.$$

Also,

$$|E'(b)| \leq C_{\#} S \lambda^{-1}$$

and

$$\begin{aligned} |E'(b) - E'(b_0)| &\leq \max_{t \in (t_{\min}, t_{\max})} |E''(t)| \cdot |b - b_0| \leq C_{\#} S \lambda^{-2} \cdot C_{\#} \lambda^{4\epsilon-2} \\ &\leq C_{\#} S \lambda^{4\epsilon-4}. \end{aligned}$$

These estimates show that

$$\begin{aligned} &\left| \frac{1}{2} E'(b) \tilde{\chi}(b) - \frac{1}{2} E'(b_0) \tilde{\chi}\left(\frac{1}{\pi} \phi(0) - \frac{1}{2}\right) \right| \\ &\leq \frac{1}{2} |E'(b)| \cdot \left| \tilde{\chi}(b) - \tilde{\chi}\left(\frac{1}{\pi} \phi(0) - \frac{1}{2}\right) \right| \\ &\quad + \frac{1}{2} |E'(b) - E'(b_0)| \cdot \left| \tilde{\chi}\left(\frac{1}{\pi} \phi(0) - \frac{1}{2}\right) \right| \\ &\leq C_{\#} S \lambda^{-1} \cdot C_{\#} \lambda^{-1} + C_{\#} S \lambda^{4\epsilon-4} \leq C_{\#} S \lambda^{-2}. \end{aligned} \quad (29)$$

Also, by definition of $\Phi(E)$ and $E(t)$, we have $E'(b_0) = (\Phi'(E)|_{E=E(b_0)})^{-1} = ((1/\pi) \phi'(0) + (1/48\pi) \psi'(0))^{-1}$ since $E(b_0) = 0$. Thus,

$$\frac{1}{2} E'(b_0) = \frac{\pi}{2} (\phi'(0))^{-1} \left(1 + \frac{1}{48} \frac{\psi'(0)}{\phi'(0)} \right)^{-1}.$$

From (3), (4), we get $|\psi'(0)/\phi'(0)| \leq C_{\#} \lambda^{-2}$, so

$$\left| \frac{1}{2} E'(b_0) - \frac{\pi}{2} (\phi'(0))^{-1} \right| \leq C_{\#} \lambda^{-2} |E'(b_0)| \leq C_{\#} \lambda^{-2} \cdot S \lambda^{-1} = C_{\#} S \lambda^{-3}.$$

Combining this with (29), we find that

$$\begin{aligned} + \frac{1}{2} E'(b) \tilde{\chi}(b) &= + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) \\ &\quad + \text{Error}_4, \quad \text{with } |\text{Error}_4| \leq C_{\#} S \lambda^{-2}. \end{aligned} \quad (30)$$

Next, we examine the term $-\frac{1}{2} E'(a) \tilde{\chi}(a)$. Since $E(a) = E_*$, we have as above

$$\begin{aligned} E'(a) &= (\Phi'(E_*))^{-1} = \left(\frac{1}{\pi} \phi'(E_*) + \frac{1}{48\pi} \psi'(E_*) \right)^{-1} \\ &= \pi (\phi'(E_*))^{-1} \left(1 + \frac{\phi'(E_*)}{48\psi'(E_*)} \right)^{-1} \end{aligned}$$

with $|\phi'(E_*)/\psi'(E_*)| \leq C_{\#} \lambda^{-2}$ by (3), (4). Therefore, $|E'(a) - \pi(\phi'(E_*))^{-1}| \leq C_{\#} \lambda^{-2} |E'(a)| \leq C_{\#} S \lambda^{-3}$, since $|E'(a)| \leq C_{\#} S \lambda^{-1}$.

We turn our attention to

$$\tilde{\chi}(a) = \tilde{\chi} \left(\frac{1}{\pi} \phi(E_*) + \frac{1}{48} \psi(E_*) - \frac{1}{2} \right) = \tilde{\chi} \left(\xi - \frac{1}{2} \right) \quad \text{with } |\xi| \leq C_{\#} \lambda^{-1}.$$

The definition of $\tilde{\chi}$ gives therefore $\tilde{\chi}(a) = \min_{k \in \mathbb{Z}} |(\xi - \frac{1}{2}) - k - \frac{1}{2}|^2 - \frac{1}{12} = \xi^2 - \frac{1}{12}$, i.e., $|\tilde{\chi}(a) + \frac{1}{12}| \leq C_{\#} \lambda^{-2}$. Thus,

$$\begin{aligned} &| -\frac{1}{2} E'(a) \tilde{\chi}(a) - (-\frac{1}{2}) (\pi(\phi'(E_*))^{-1}) \cdot (-\frac{1}{12}) | \\ &\leq \frac{1}{2} |E'(a)| \cdot |\tilde{\chi}(a) + \frac{1}{12}| + \frac{1}{2} \cdot (\frac{1}{12}) \cdot |E'(a) - \pi(\phi'(E_*))^{-1}| \\ &\leq C_{\#} S \lambda^{-1} \cdot C_{\#} \lambda^{-2} + C_{\#} S \lambda^{-3} \leq C'_{\#} S \lambda^{-3}. \end{aligned}$$

So

$$-\frac{1}{2} E'(a) \tilde{\chi}(a) = + \frac{\pi}{24} (\phi'(E_*))^{-1} + \text{Error}_5, \quad \text{with } |\text{Error}_5| < C_{\#} S \lambda^{-3}. \quad (31)$$

Now we can substitute (25), (26), (27), (28), (30), (31), into (23), (24) to obtain the following result.

$$\begin{aligned} \text{sneg}(H) = & \int_{a_0}^{b_0} E(t) dt + \frac{E_*}{48\pi} \psi(E_*) + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) \\ & + \frac{\pi}{24} (\phi'(E_*))^{-1} + \text{Error}_6, \quad \text{with } |\text{Error}_6| \leq C_* S \lambda^{2\varepsilon-2}. \end{aligned} \quad (32)$$

Next, we evaluate the integral in (32). The computation of the corresponding integral in the section Truncated Eigenvalue Sums carries over here word for word, to tell us once more that

$$\begin{aligned} \int_{a_0}^{b_0} E(t) dt = & -\frac{E_*}{\pi} \phi(E_*) - \frac{E_*}{48\pi} \psi(E_*) \\ & - \frac{2}{3\pi} \int_{I_{\text{BVP}}} \left(-V(x) \right)_+^{3/2} dx + \frac{2}{3\pi} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx \\ & + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) (-V(x))_+^{-1/2} dx \\ & - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx. \end{aligned}$$

Putting this into (32), we find that

$$\begin{aligned} \text{sneg}(H) = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ & + \frac{2}{3\pi} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx \\ & + \frac{\pi}{24} (\phi'(E_*))^{-1} - \frac{E_*}{\pi} \phi(E_*) + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) \\ & + \text{Error}_6, \end{aligned} \quad (33)$$

with $|\text{Error}_6| \leq C_* S \lambda^{2\varepsilon-2}$.

Next, we let E_* tend to $V(x_0)$ and pass to the limit in (33). As in the proof of the First WKB Eigenvalue Sum Theorem, we have

$$\begin{aligned} \lim_{E_* \searrow V(x_0)} \int_{I_{\text{BVP}}} (E_* - V(x))_+^{3/2} dx &= 0, \\ \lim_{E_* \searrow V(x_0)} \frac{E_*}{\pi} \phi(E_*) &= 0, \end{aligned}$$

$$\lim_{E_* \searrow V(x_0)} \left\{ \frac{\pi}{24} (\phi'(E_*))^{-1} - \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (E_* - V(x))_+^{-1/2} dx \right\} = 0.$$

Using these results to pass to the limit in (33), we obtain the following theorem.

THIRD WKB EIGENVALUE SUM THEOREM. *Suppose we are in the setting of the WKB Theorem on Low Eigenvalues, with $E_\infty = 0$. Assume hypotheses (H0*)–(H6*) of that theorem, and assume also $-\lambda^{-3\epsilon}S < V(x_0) < 0$. Then the sum of the negative eigenvalues of $H = -d^2/dx^2 + V(x)$ on I_{BVP} , with Dirichlet or Neumann boundary conditions, is given by*

$$\begin{aligned} \text{sneg}(H) = & -\frac{2}{3\pi} \int_{I_{\text{BVP}}} (-V(x))_+^{3/2} dx + \frac{1}{24\pi} \int_{I_{\text{BVP}}} V''(x) \cdot (-V(x))_+^{-1/2} dx \\ & + \frac{\pi}{2} (\phi'(0))^{-1} \tilde{\chi} \left(\frac{1}{\pi} \phi(0) - \frac{1}{2} \right) + \text{Error}, \end{aligned}$$

with $|\text{Error}| \leq \lambda^{5\epsilon-2}S$.

Recall that $\phi(E) = \int_{I_{\text{BVP}}} (E - V(x))_+^{1/2} dx$, $\phi'(E) = \frac{1}{2} \int_{I_{\text{BVP}}} (E - V(x))_+^{-1/2} dx$, and $\tilde{\chi}(t) = \min_{k \in \mathbb{Z}} |t - k - \frac{1}{2}|^2 - \frac{1}{12}$.

ACKNOWLEDGMENT

We are grateful to Maureen Schupsky for her skill and patience in Texting our manuscript in difficult circumstances.

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